search for ore or oil; namely, 1) Magnetic, 2) Electrical, 3) Electro-magnetic, 4) Gravitational, 5) Seismic. Of these five methods, two, viz., the magnetic and gravitational, are pure receiving methods in that nature always furnishes the messengers and we need only listen. Two, viz., the electromagnetic and the seismic, are sending-receiving methods, the messengers being furnished by us. The electrical method makes use of both types of messengers.

B. The Methods of Geophysical Exploration for Ore and Oil

1. *The magnetic method.* In order to appreciate the significance of the mathematical problems, it is necessary to have at least a general idea of these five geophysical methods. Let us consider, first, the magnetic method, which, although it is particularly simple, is in several regards typical of all the methods which make use of electrical effects. The magnetic method of surveying is based on two simple facts; viz., the fact that a magnet attracts or repels a compass needle; and the fact that certain substances, when placed in a magnetic field, automatically become magnetized by that field and hence become magnets. When various magnetizable substances are located in the same magnetic field, the intensities of the magnetism induced in the various substances are different, and the ones that magnetize more strongly are said to possess the greater magnetic susceptibility. Granted the two effects mentioned just above, it is only necessary to have some external magnetic field present in order that the various rock or mineral masses become magnetized in amounts depending upon their various susceptibilities. We can then observe the action on a compass needle of the induced magnets thus formed, and can draw inferences as to the location, size, and constitution of the active masses. The necessary external magnetic field is conveniently omnipresent, for the earth's own magnetic field exists not only in the air, but at all depths below the surface, and plays this role very satisfactorily. This inducing external magnetic field furnishes the messengers, penetrating to the unknown regions, which stimulate or excite these unknown regions to send to us further messengers (the induced magnetic field) whose reports we receive in our instruments and interpret through our theory.

Now the vast majority of rocks and minerals have very low susceptibilities. That is to say, they are magnetized exceedingly feebly when placed in a magnetic field. Certain of the iron compounds, on the other hand, have abnormally high values for their susceptibilities. The most striking and important example is magnetite, which has a susceptibility ranging from a hundred to a thousand or more times the minimum detectable value. Besides magnetite, ferriferous minerals such as franklenite, ilmenite, pyrrhotite, and, to a lesser degree, specular hematite, are all strongly magnetic. It is not correct, however, to assume that the magnetic method is restricted to the search for iron ore. In a copper region, for example, it may be of the greatest importance to be able to locate and trace the contact between the sedimentary rock and an igneous intrusion; and there is often sufficient difference in susceptibility to permit

this. Magnetic methods have, in fact, been used to locate inclusions of iron formation in an intrusive gabbro; to locate faults in iron formations; to map distributions of lavas and intrusives; to trace copper bearing lodes; to trace

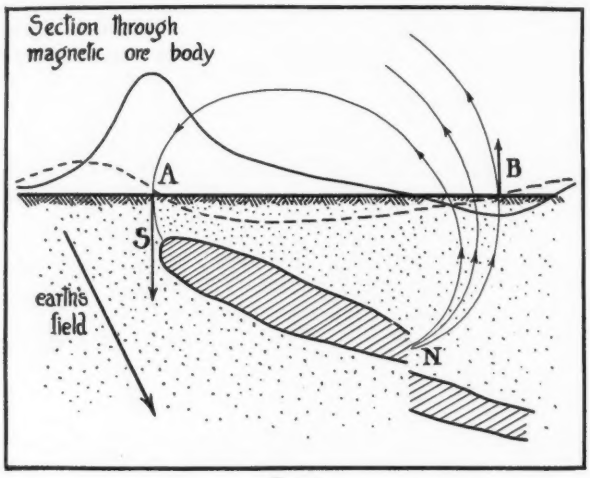


FIG. 1

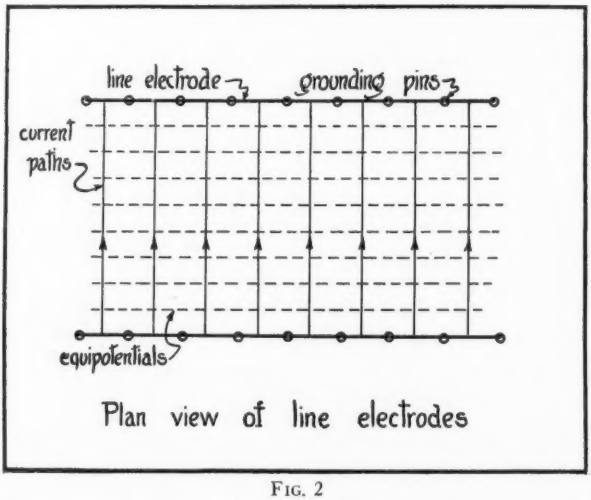


FIG. 2

igneous contacts and dikes; to trace the outcrop of a sedimentary bed; to locate gold placer deposits; and in many other connections as well.

We may now trace in a little more detail how the magnetic method pro-

[2] Figures 1 and 7 are modifications of figures occurring in Eve and Keyes *Applied Geophysics.*

ceeds. Suppose the cross hatched region of figure[2] 1 represents a magnetic ore body while the surrounded dotted material is relatively inert. Due to the earth's magnetic field, indicated by the arrow, the ore body becomes magnetized. If the ore body were absent, the magnetic field of the earth would be sensibly uniform over the region in question. Due to the ore body, the field is distorted. Any instrument measures the entire field, composed of the normal earth's field plus the field due to the ore body, If, however, one subtracts from the total measured field the normal earth's field, he obtains the portion due to the ore body itself. This abnormal part of the field, due to the disturbing presence of the body, is called the magnetic anomaly; and since it is a force, one may resolve it into a horizontal anomaly and a vertical anomaly. The solid and dotted curves, drawn with the ground line as the axis of abscissas, show the variation of the vertical and horizontal magnetic anomalies. The light curved lines passing from the north to the south poles of the ore magnet roughly indicate the field of magnetic force due to the ore magnet. For example, at a point A approximately above the south pole of the ore magnet, the field of the ore magnet is directly downward, so that the horizontal anomaly is zero, while the vertical anomaly is large. A similar situation exists at B over the north pole except that the vertical anomaly is reversed in sign. The fact that the faulted end N of the ore body is more deeply buried is revealed by the fact that the vertical anomaly at B is considerably less than that at A. If the indicated fault were not present so that the north pole of the ore magnet were very deeply buried, the negative value of the vertical anomaly would be entirely absent. It is clear that the quantitative interpretation of such data involves a large amount of theoretical information concerning the exact way in which bodies of various shapes magnetize, and concerning the fields which they then cause.

The actual measurement of the intensity and direction of the magnetic field at any point involves the use of instruments such as the dip needle, the magnetometer, the magnetic variometer, the earth inductor etc. It is outside our purpose to describe these instruments or explain their use. As a concluding remark concerning the magnetic method, it should be emphasized that this method reveals discontinuity of substance or structure only when these discontinuities are characterized by different values of the magnetic susceptibility. Only materials which have reasonably differing susceptibilities can be distinguished.

2. *The electrical method.* When a source of electric current exists in any three-dimensional region, the spatial distribution of the resulting currents depends on the geometry of the region, the location and shape of the source, and upon the electrical conductivities of the various parts of the region. If the region be homogeneous as regards conductivity, one may call the accompanying current distribution a *normal* one. If there be any departures from a uniform conductivity, the current distribution is abnormal. Accompanying this abnormal distribution of current is an abnormal distribution, throughout the interior and on the surface of the region, of the electrical potential. The electrical

methods of surveying measure the potential at points on the surface of the ground, and deduce from abnormalities in these potentials the sub-surface variation in electrical conductivities. As a source of electric current, one uses

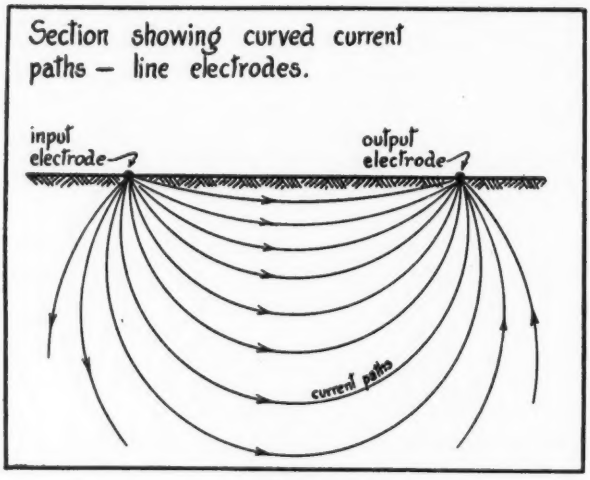


FIG. 3

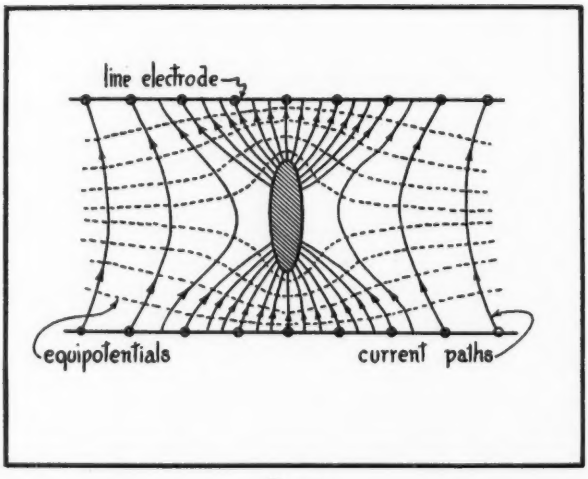


FIG. 4

either point or line electrodes, maintained at desired potentials by connecting them to a battery or generator. (See Fig. 2.) Long line electrodes are usually approximated by connecting together a series of metal stakes or grounding pins spaced at regular intervals. In the case of a homogeneous earth, the shallow

portion of the current which stays near the surface of the ground passes on straight lines from one electrode to the other, the equi-potentials being straight lines parallel to the line electrodes. It is only the surface current that passes along the straight line shown. In general, the current passes along the curved lines shown in Fig. 3. If there is in the region between the two line electrodes a mass of higher conductivity, then the whole current-distribution is altered. Fig. 4 shows the current paths and the accompanying equi-potentials, when a highly conducting lens is located between the electrodes. The distortion of the normally straight and parallel equi-potentials reveals the presence of the mass of higher conductivity.

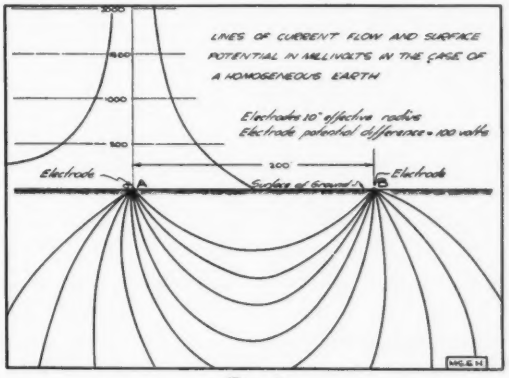

LINES OF CURRENT FLOW AND SURFACE POTENTIAL IN MILLIVOLTS IN THE CASE OF A HOMOGENEOUS EARTH
Electrodes 10" effective radius
Electrode potential difference = 100 volts
Electrode
Surface of Ground
Electrode
M.G.&H.

FIG. 5

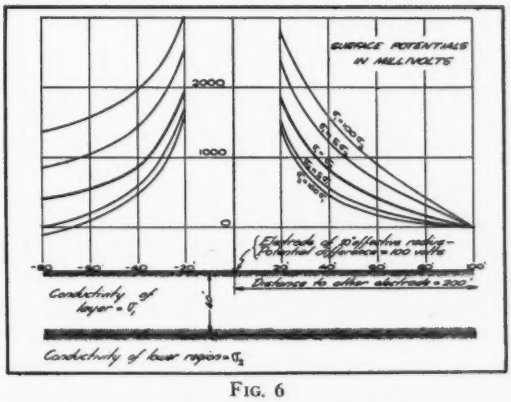


FIG. 6

The point electrode method differs from the line electrode method principally in the fact that the current enters and leaves the earth at points, rather than along lines. Fig.[3] 5 shows in section the surface of the ground; the input

[3] Figures 5 and 6 are the property of Mason, Slichter, and Gauld, Madison, Wisconsin, and are used with their permission.

and output electrodes at A and B; the deep-lying current paths; and, plotted above the surface of the ground, the potential at the surface of the ground and along the line joining the two electrodes. Fig. 6 illustrates the point electrode method as applied to the location of a layer of depth 40 feet and one conductivity σ_1, underlaid by material of conductivity σ_2. The heavy curve $\sigma_1=\sigma_2$ is the normal potential found when the layer is absent. If the underlying region is a good conductor, the potential curve falls below the normal curve; while if the layer is much the better conductor, the curve lies above the normal curve.

3. *The electromagnetic method.* One should note that the magnetic and the electric methods utilize different physical characteristics of rock or mineral. The success of one method depends on differences in magnetic susceptibilities; the other on differences in electrical conductivity. When one shouts down to the ore, he must be careful to ask questions that are suited to the particular ore being sought. One characteristic of the electromagnetic method (which we will now consider briefly) is that its questions and answers do not depend on one single electrical characteristic but rather on several. This is both an advantage and a drawback. When one shouts out a question in five languages, there is, to be sure, a good chance that some one will understand and reply; but there is also a good chance that one will get several simultaneous and hence confused replies.

The electromagnetic method is not one that lends itself well to simple explanation. The sending device is usually a coil or loop antennae, a good deal like those sometimes used in radio. The receiving device is similar to a radio receiving set. Through the sending coil is passed an alternating current. This alternating current sets up, for very considerable distances from it, both an electric and a magnetic field. These fields penetrate the earth, the effective depth of penetration depending on the strength of the source, the frequency of the alternating current, and on the electrical and magnetic properties of the earth and rocks. Due to these penetrating fields there are induced, in the earth and in all bodies present, electric currents, magnetizations, and electric polarizations. Thus the total effect at any point is due, in various parts, to the electrical conductivities, the magnetic susceptibilities, and the dielectric properties of the earth and rocks. There are, also, very many things one can and should measure with his receiving instruments before he can hope to understand the messages the hidden region is trying to send. One can measure the magnitude and direction of the electric field and of the magnetic field; he can measure the phase difference, between these two fields, at various points; and he can observe the dependence of these quantities on variation of the frequency of the alternating current. One obviously has, here, a much more complicated situation than existed in the magnetic or electric cases, where only one quantity was available for measurement.

4. *The gravitational method.* All of the methods so far considered make use of the fact that various rocks and minerals have varying electrical properties. The gravitational method, as one might expect, has nothing to do with electrical effects, and depends solely upon density. The gravitational method makes use of almost unbelievably delicate instruments, called torsion balances. In essence, such an instrument is very simple. Two equal masses (Fig. 7) m and m' are swung, at different levels, by a very fine wire and aluminum rod system. A heavy mass below this torsion balance and on the side towards the reader attracts the lower bob more than the upper (simply because it is nearer the lower). This difference in attraction results in a torque which turns the rod

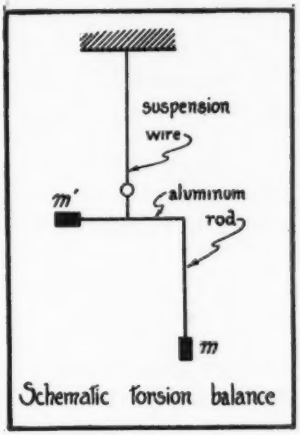


FIG. 7

and mass system until a counter-torque is developed in the wire suspension. In this way regions of sub or super-normal density below and to one side of the balance cause deflections of the suspended system.

5. *The seismic method.* We turn now to the last of the five methods, the seismic. In this method a powerful source of vibrational waves is obtained by exploding a buried charge of TNT. These waves are somewhat like the electric currents flowing between power electrodes, in that the vibrational waves also travel on curved paths, penetrate to very considerable depths, and eventually emerge at the surface. These waves are received by seismographs, entirely similar to those used to record earthquakes. These wave messengers are thus sent down, and return to give us their messages. And we draw our inferences chiefly from the time at which the messengers return to us; for the speeds with which they travel depend on the densities and elastic properties of the materials through which they have passed. The seismic method has been used chiefly in oil work, although it has also been used, in Sweden, to determine depth of overburden. The seismic method depends on several mechanical properties

of the rocks, rather than merely on density, as does the gravitation. Unfortunately we have meager data concerning the Young's modulus, bulk modulus, shear modulus, and Poisson's ratio for earth, rocks, and minerals, and the theoretical aspects of this method are by no means complete.[4]

C. The Mathematical Problems

Unsatisfactory as so brief a resume must necessarily be, we turn now to a consideration of the mathematical problems connected with the geophysical methods of prospecting. Many of these problems are new problems; not new in the sense that they involve recent theories, but new in that they involve, for the most part, the application of classical theories to hitherto uninteresting situations.

Until the advent of radio, the whole great development of the electrical art concerned the use of electricity *in wires*. A wire is essentially a one-dimensional object, its cross sectional dimensions being negligible as compared to its length, so that the theory of electricity in wires is essentially a one-dimensional theory. The problem of the spreading of radio signals was one of the first really practical electrical problems which involved more than one dimension. The theoretical problems of geophysical prospecting are practically all either two or three dimensional problems. A one-dimensional problem means, to the mathematician, an ordinary differential equation to solve; two or three-dimensional problems mean partial differential equations. We are all well aware of the enormous increase of difficulty that results, in boundary value problems, when the number of dimensions is increased from one to two, or from two to three. A startlingly small number of three-dimensional problems have been solved in exact form; and of the exact solutions actually obtained, few are usable in a practical computative sense.

The unity, from the point of view of mathematical theory, of all of the methods of geophysical prospecting becomes evident when we consider the fundamental equations which govern the various phenomena. One may write the general wave equation,

$$\nabla^2\Psi - \frac{1}{\kappa^2}\frac{\partial^2\Psi}{\partial t^2} = -\Theta,$$

and the equation suitable for any one of the five methods may then be obtained by specializing the quantities Ψ, κ, and Θ. In general, Ψ is the quantity whose behavior is sought; κ is the velocity with which values of Ψ are propagated,

[4] The reader who wishes to supplement these exceedingly brief remarks concerning the magnetic, electrical, electromagnetic, gravitational, and seismic methods is referred to the volume, *Geophysical Prospecting*, published in 1929 by the American Institute of Mining and Metallurgical Engineers, and in particular to the article *Geophysical Prospecting For Ores*, by Dr. Max Mason (also published in the Bulletin of the Mining and Metallurgical Society of America, vol. 20 (1927), pp. 93–133); also to the bibliography by C. A. Heiland, published in the *Annotated Bibliography for Economic Geology* for 1928.

and Θ is a sort of forcing function or source function which generates non-vanishing values of Ψ. Thus, for the magnetic method:

Ψ is the magnetic vector potential $\boldsymbol{A}$;
κ^2 is infinite so that $(1/\kappa^2)\ \partial^2\boldsymbol{A}/\partial t^2$ is zero;
Θ is zero;

and the equation becomes Laplace's equation

$$\nabla^2\boldsymbol{A} = 0.$$

For the electric method:

Ψ is the electric potential Φ;
κ^2 is infinite;
Θ is zero;

and the equation becomes Laplace's equation

$$\nabla^2\Phi = 0.$$

For the electro magnetic method:

Ψ is the electric vector $\boldsymbol{E}$,
or the magnetic vector $\boldsymbol{B}$;
κ^2 is $\mu\epsilon/c^2$;
Θ is $-(\nabla\rho/\epsilon)-(\mu/c^2)\ \partial(\rho\boldsymbol{u})/\partial t$,
or μ curl $(\rho\boldsymbol{u})/c$,

where μ is the magnetic permeability, ϵ the dielectric constant, c the velocity of light in free space, ρ the density of charge, and $\boldsymbol{u}$ the vector velocity of charge. The general equation thus gives the two forms,

$$\nabla^2\boldsymbol{E} - \frac{\mu\epsilon}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2} = \frac{1}{\epsilon}\nabla\rho + \frac{\mu}{c^2}\frac{\partial(\rho\boldsymbol{u})}{\partial t},$$

$$\nabla^2\boldsymbol{B} - \frac{\mu\epsilon}{c^2}\frac{\partial^2\boldsymbol{B}}{\partial t^2} = -\frac{\mu}{c}\,\text{curl}\,(\rho\boldsymbol{u}).$$

For the gravitational method:

Ψ is the gravitational potential Φ;
κ^2 is infinite;
Θ is the volume density of matter ρ;

and the equation becomes Poisson's equation

$$\nabla^2\Phi = -\rho.$$

For the seismic method:

Ψ is the dilatation s;
κ^2 is $(\lambda+2\mu)/\rho$;
Θ is zero;

where λ and μ are the standard elastic constants, and ρ is the density. The general wave equation then becomes

$$\nabla^2 s - \frac{\rho}{\lambda+2\mu}\frac{\partial^2 s}{\partial t^2} = 0.$$

One is not, of course, surprised to learn that the same general differential equation governs all these methods. The wave equation, and as special cases of it, the equations of Laplace and Poisson, express to the mathematician the very essence of the manner in which natural phenomena proceed; this observation having increased significance now that we have a wave theory basis for particle and quantum dynamics. The observation that there is a central unifying structure behind all these methods may often assist one in adapting one mathematical solution to several of the methods. One should remember that the formal equivalence of two such problems, however, involves equivalent boundary conditions, as well as the same differential equation. Let us, then, note the types of boundary conditions which arise, restricting our further attention to the three methods which make use of electrical effects. In the magnetic method:

$\boldsymbol{A}$ is continuous everywhere.
$\boldsymbol{A}$ vanishes at infinity as $1/R$.
The tangential derivative of $\boldsymbol{A}$ is continuous.
The normal derivative of $\boldsymbol{A}$ is discontinuous.

$$\frac{\partial \boldsymbol{A}}{\partial n_1} + \frac{\partial \boldsymbol{A}}{\partial n_2} = \left(1 - \frac{1}{\mu}\right)[\boldsymbol{n}, \operatorname{curl} \boldsymbol{A}].$$

For the electrical methods:

Φ is continuous everywhere.
Φ vanishes at infinity as $1/R$.

$$\sigma_1 \frac{\partial \Phi}{\partial n_1} + \sigma_2 \frac{\partial \Phi}{\partial n_2} = 0.$$

For the electromagnetic methods:

$\boldsymbol{E}$ and $\boldsymbol{B}$ vanish at infinity as $1/R^2$.
The tangential components of $\boldsymbol{E}$ are continuous.
The normal component of $\boldsymbol{B}$ is continuous.

$$\mu_2 B_{t_1} + \mu_1 B_{t_2} = 0.$$
$$\epsilon_1 E_{n_1} + \epsilon_2 E_{n_2} = 0.$$

Thus in all these cases, the boundary equations which hold across a surface separating two media are linear in the dependent variable or its normal derivatives, the coefficients of these linear relationships characterizing the electrical properties of the two media.

Such boundary value problems have received the attention of mathematicians for a century and more. But the problems of geophysics differ from many classical problems in several respects. First of all, as mentioned above,

they are two or three-dimensional problems. Secondly, there are practically always three important regions of space involved, each with its own characteristic electrical properties,—the air, the normal homogeneous earth, and the ore mass. The older problems seldom involved more than two distinguishable regions.[5] Thirdly, it is necessary to have exact knowledge near the sources of disturbance, rather than far away, as in the classical problem of radio signals. Fourthly, it is useful, in the electromagnetic case, to obtain solutions when the wave length is comparable with the dimensions of the diffracting ore mass, rather than small, as in the classical optical case. This consideration makes invalid many of the approximation methods used in earlier work. Fifthly, it is necessary, in general, to assign finite but different conductivities to the various regions under consideration. This is a point of great importance, and the source of much difficulty. Sixthly, many regions of geometrical shapes, which were previously of only academic interest, now become of great practical interest. And, lastly, it is essential now that the solutions be usable from a computative point of view.

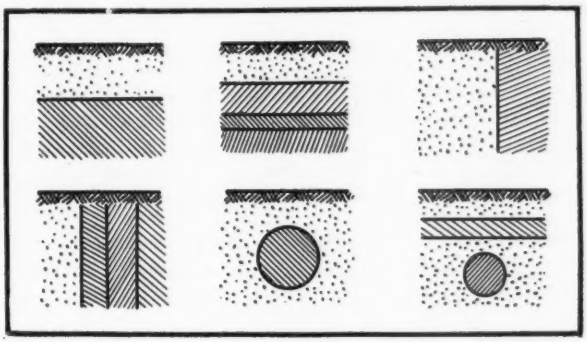


FIG. 8

The actual problems of nature are impossibly complicated by minor non-homogeneities, by uneven surface topography, by irregular moisture content, by the possible presence of several disturbing masses, and by many other things

[5] One usually avoids the necessity of satisfying the boundary conditions on the surface of the ground between the earth and air by reflecting the entire configuration in the surface of the ground. Thus rather than dealing with a problem which involves the air, a semi-infinite earth, and a buried sphere, one solves the problem of an infinite earth containing two spheres. The plane which is perpendicular, at its mid-point, to the line joining the centers of the two spheres is a plane of symmetry. If the current source be located on this plane, no current will cross this plane. Thus one can "throw away" the upper half of the solution, and retain the correct solution for a semi-infinite earth and a buried sphere. This reflection method does not, however, afford an essential simplification of the three-region difficulty mentioned above: for there are still three regions,—the infinite earth, the first sphere, and the second sphere. Taking into account the interaction between the two spheres, is exactly equivalent mathematically to taking account of the boundary conditions between the earth and the air.

as well. Just as a child must begin with simple tunes if he is ever to understand a symphony, so the geophysicist must, at first, be content with simplifications and over-idealizations of his problems. Thus, for the electrical method, one attempts to calculate the distribution of currents when a homogeneous earth is overlaid by one or more plane layers of different conductivities; when there are one or more parallel vertical planes across which the conductivity suddenly changes; when there is a submerged vertical dyke, or a submerged ledge; or when an otherwise homogeneous earth contains a buried sphere or ellipsoid.

Figure 8 shows, schematically, cases of current flow which have been satisfactorily solved both for point and long lines electrodes. The dotted region indicates the normal earth, while the cross hatched regions indicate ore bodies or structures of differing conductivities. The layered structures have been treated by several investigators although, for commercial reasons, a complete analysis has not been published.[6] The layered structure containing a buried sphere has been treated by the writer.[7] The point electrode in the presence of a buried sphere has recently been solved exactly.[8] Similar solutions for a buried ellipsoid would be of great usefulness since an ellipsoid can take on, as special cases, long needle like or flat lens like shapes.

The mathematical analysis involved in the layered structure was indicated by Maxwell;[9] and, for the simpler case of infinite conductivity ratio, by Riemann.[10] This analysis involves infinite series of the type

$$\sum_{-\infty}^{+\infty}{}_n(-1)^n\left\{[\rho^2+(z+2nh-a)^2]^{-1/2}-[\rho^2+(z+2nh+a)^2]^{-1/2}\right\}$$

which are rather slowly convergent and very tedious to compute. In this expression z measures distance perpendicular to one of the horizontal planes bounding the layer, $\rho^2=x^2+y^2$, where x and y are Cartesian coordinates in a horizontal plane, h is the thickness of the layer, and a is the spacing between electrodes. These series have been satisfactorily handled for large ρ, but their study for small ρ is a pressing problem.

The current flow problems, in the case of long line electrodes and suitable non-homogeneities, are two-dimensional problems involving the solution of the Laplace equation $\nabla^2\Phi=0$. Here the method of conjugate functions applies. Much that has been written concerning the theory of conformal mapping has come from the pens of mathematicians who had little interest or information about the application of this theory to electrical problems, or from the pens of physicists who had only a mild respect or appreciation for the mathematical

[6] See, however, J. N. Hummel, *Der Scheinbare Spezifische Widerstand bei Vier Planparallelen Schichten*, Zeitschrift für Geophysik, vol. 5 (1929), Heft 5/6, 5–228.

[7] Not published in full. See, however, *Certain Applications of the Surface Potential Method*, American Institute of Mining and Metallurgical Engineers, Tech. Pub. 121 (1928).

[8] Unpublished thesis by Dr. J. H. Webb, now of Williams College.

[9] James Clerk Maxwell, *Electricity and Magnetism*, 1873, Vol. I, § 317, page 367.

[10] Riemann, *Gesammelte Werke*, page 59.

aspects. We need an accurate and illuminating exposition of the Schwartz-Liouville theory of the mapping of polygons and of the application of this theory to electrical problems, along with an encyclopaedic collection of examples.

There are many simple shapes of ore mass for which the solution is needed. Fig. 9 indicates six of these unsolved problems, the cross hatched regions again

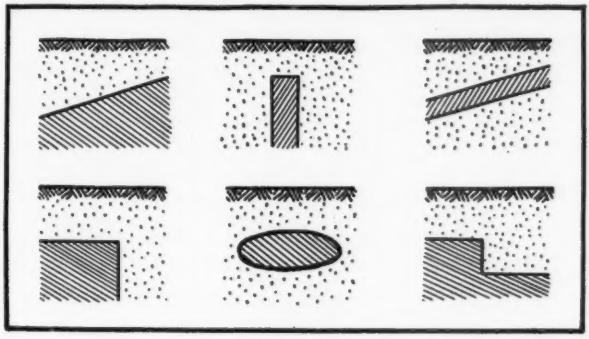


FIG. 9

being the ore masses of higher conductivity, and the dotted regions being the normal earth. One desires the current flow, in all these cases, whether the current enters the earth at a point electrode, or along a line electrode. It is interesting to note that although the horizontal layer has been treated rather fully, the slanting layer has so far withstood all attempts at solution. The ordinary method of images is not applicable to the slanting layer.[11]

It would also be of great interest to treat some simple cases of non-uniform but continuously changing conductivity. If the conductivity is variable, the governing differential equation is no longer linear. Such a problem was formerly not of much practical interest, but is now exceedingly interesting. The most promising mode of attack seems to be through an integral equation set-up, treating the continuously changing density as a limiting case of multi-layered structures.

The electromagnetic method wishes to know, as a first simple approximation, the induced currents and the total earth and air fields, when one uses, as a sending device, an oscillating electric or magnet dipole, or a long overhead wire carrying alternating current. This problem is fairly solved for the case of a homogeneous plane earth. And we have fairly satisfactory approximations to cover the case of a sphere buried in an otherwise homogeneous plane earth. The long line over a layer of finite thickness and conductivity has just been solved.[12] A dipole over such a layer has not yet been solved, at least in usable form.[13] In

[11] See, however, the last footnote of the paper.

[12] Doctor's dissertation, as yet unpublished, by Dr. H. P. Evans, University of Wisconsin.

[13] Since this paper was written, this problem has been solved. See *Asymptotic dipole radiation formulas*, by W. Howard Wise, in the Bell System Technical Journal, VIII, 4, page 662 (October 1929).

fact, a doublet over an infinitely thin layer was not treated until 1922 when Max Abraham, in the last paper he wrote,[14] calculated the induced currents for this case, but made an error in the calculation of the fields. A doublet located over a homogeneous earth in which is buried an *ellipsoidal* mass presents a problem of great interest, and one which has not been usefully solved. These problems of the induced fields in the presence of oscillating dipoles are analogous to the classical problems of the diffraction of light by an obstacle. They are more difficult, however, since the various media have *finite*, but differing, conductivities; and also since they are diffraction problems of the Fresnel, rather than the more usual Fraunhofer type—that is, the source is at a finite distance rather than at infinity. One realizes the probable difficulty of these geophysical problems when he remembers that no one of the classical diffraction problems was solved exactly until Sommerfeld obtained the solution for the straight edge.[15] Any mathematician would do a real service who would study and extend the Sommerfeld multiple-space reflection method, carrying it through for finite conductivities, using sources at finite distances, and, most important of all, discovering how these complicated formal solutions can be made available for computation.

D. Conclusions

These scattered and hasty remarks can hardly be expected to give more than a shadowy suggestion of the sort of problems that confront the theoretical geophysicist. The ancient searcher after ore sought to find the hidden treasures of the earth by using a forked hazel stick, whose magic properties would direct him to the buried veins. The geophysicist of today, with his dip-needle, his volt-meter, his radio apparatus, his slide-rule, and his text-book on partial differential equations, is but the modern counterpart of the old practicer of the black arts who went about with his forked hazel wand. Many years ago when his international fame rested solely on his ability as a mining engineer, President Herbert Hoover, together with Mrs. Hoover, translated from the Latin a very famous old treatise on mining, written by Georgius Agricola during the twenty years preceding its publication in 1566. This book contains the first description of the application of divining rods to the location of ore. The last figure (Fig. 10) shows a wood-cut occurring in this interesting (and now rare) book. You will observe the various experts going about with their hazel wands, and the shallow trenches which are being dug to test the indications. With a freedom from superstition that seems amazing for his age, and with a scientific clarity that would grace any age, Agricola considers the evidence for and against the forked twig, and concludes "Therefore a miner, since we think he ought to be a good and serious man, should not make use of an enchanted twig, because if he is prudent and skilled in the natural signs, he understands that a forked stick is

[14] M. Abraham, Zeitschrift für Angewandte Mathematik und Mechanik, 1922, page 109.

[15] Riemann-Weber, *Die Differential-und Integralgleichungen der Mechanik und Physik*, 1927, Vol. 2, page 433.

of no use to him." We must not hastily judge, however, that Agricola would condemn the modern forked stick. His truly scientific spirit, and the attitude

FIG. 10

proper for us, are alike indicated by this ancient author's remark: "Since this matter remains in dispute and causes much dissension amongst miners, I consider it ought to be examined on its own merits."

AVERAGE CURVES AND THEIR HARMONIC ANALYSIS

By LOUIS BRAND, University of Cincinnati

This paper gives an analytic solution of an important problem arising in the design of alternating current generators. My colleague, Professor W. C. Osterbrock of the department of electrical engineering, states the problem as follows:

"In the alternating current generator magnetic flux is set up by a revolving field structure excited from a separate source by a continuous current. The electromotive force is generated in coils distributed in slots on a stationary armature. The value of the electromotive force in any conductor at any time is $e = Blv$ electrostatic units, where B is the magnetic flux density at the conductor (gauss), l the length of the conductor (cm.), and v the peripheral velocity of the field structure (cm./sec.). This will be a function of period 2π of the angular displacement along the armature such that

$$f(x + \pi) = f(x - \pi) = -f(x). \tag{I}$$

"The various coils forming one group are distributed in slots covering one pole pitch (π radians) and are connected in series so that their electromotive forces add. The problem then arises to find the sum of n such functions as given above, each displaced from the preceding by an interval π/n."

This problem is solved graphically by drawing the original *base curve* $y=f(x)$ and its successive displaced positions and adding the ordinates at any point to obtain the sum curve. However the problem may be approached analytically by finding the limit of the average ordinate at any point as n become infinite. The ordinates of this *limit curve*, $y=F(x)$, multiplied by n, will then give a satisfactory approximation to the sum curve for a finite n if n is not too small.

When the base curve $y=f(x)$ is shifted a distance π to the right, all the ordinates between $x-\pi$ and x pass over the point x. The limit curve is clearly the average of these ordinates; hence

$$F(x) = \frac{1}{\pi}\int_{x-\pi}^{x} f(u)du. \tag{1}$$

Consider first the case when $f(x)$ satisfies the condition (I). Then

$$\pi F(x) = \int_{x-\pi}^{0} f(u)du + \int_{0}^{x} f(u)du,$$

in which the first integral equals

$$\int_{x}^{\pi} f(v-\pi)dv = \int_{\pi}^{x} f(u)du = \int_{0}^{x} f(u)du - \int_{0}^{\pi} f(u)du.$$

Hence, on writing

$$h = \frac{1}{\pi}\int_{0}^{\pi} f(u)du,$$

$$F(x) = \frac{2}{\pi}\int_{0}^{x} f(u)du - h. \tag{2}$$

From (2) we see that

$$F(0) = -h, \qquad F(\pi) = F(-\pi) = h; \tag{3}$$

$$F(x+\pi) = \frac{2}{\pi}\int_{0}^{\pi} f(u)du + \frac{2}{\pi}\int_{\pi}^{x+\pi} f(u)du - h = h - \frac{2}{\pi}\int_{0}^{x} f(u)du;$$

$$F(x+\pi) = -F(x). \tag{4}$$

Hence $F(x)$ also has the property (I). Moreover

$$F'(x) = 2f(x)/\pi. \tag{5}$$

From the standpoint of generator design it is important to have an harmonic

analysis of $F(x)$. In view of (I) the Fourier expansion of $f(x)$ in the interval $(-\pi, \pi)$ contains only the sines and cosines of odd multiples of x:

$$f(x) \sim \sum_{k=0}^{\infty} [a_{2k+1} \cos (2k+1)x + b_{2k+1} \sin (2k+1)x],$$

where

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx, \qquad b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx.$$

From (4) we see that the expansion of $F(x)$ has precisely the same form. Moreover on integrating by parts and making use of (3) and (5) we find

$$A_n = \frac{2}{\pi} \int_0^{\pi} F(x) \cos nx dx = -\frac{4}{\pi^2 n} \int_0^{\pi} f(x) \sin nx dx = -\frac{2}{\pi n} b_n,$$

$$B_n = \frac{2}{\pi} \int_0^{\pi} F(x) \sin nx dx = \frac{4}{\pi^2 n} \int_0^{\pi} f(x) \cos nx dx = \frac{2}{\pi n} a_n.$$

Therefore

$$F(x) \sim \frac{2}{\pi} \sum_{k=0}^{\infty} \left(\frac{a_{2k+1}}{2k+1} \sin (2k+1)x - \frac{b_{2k+1}}{2k+1} \cos (2k+1)x \right).$$

Owing to the factor $1/(2k+1)$, the upper harmonics are far less important in $F(x)$ than in $f(x)$. This, of course, is very desirable when the objective is a pure sinusoidal electromotive force.

We consider next the case when $f(x)$ is an odd function of period 2π:

(II) $$f(-x) = -f(x).$$

From (1) and (II) we now have

$$\pi F(x) = \int_{x-\pi}^{0} f(u)du + \int_0^x f(u)du = \int_0^x f(u)du - \int_{\pi-x}^{0} f(-v)dv,$$

(6) $$F(x) = -\frac{1}{\pi} \int_x^{\pi-x} f(u)du.$$

Hence

(7) $$F(0) = -h, \qquad F(\pi) = F(-\pi) = h;$$

$$F(-x) = -\frac{1}{\pi} \int_{-x}^{\pi+x} f(u)du = \frac{1}{\pi} \int_x^{-\pi-x} f(-v)dv = -\frac{1}{\pi} \int_x^{-\pi-x} f(u)du;$$

and, on adding

$$-\frac{1}{\pi} \int_{-\pi-x}^{\pi-x} f(u)du = 0$$

to the last integral, we find that

$$F(-x) = F(x). \tag{8}$$

Moreover,

$$F'(x) = \frac{f(x) + f(\pi - x)}{\pi}. \tag{9}$$

Since $f(x)$ is an odd function and $F(x)$ an even function, their Fourier expansions are sine and cosine series respectively:

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx, \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx dx,$$

$$F(x) \sim \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos nx, \quad A_n = \frac{2}{\pi}\int_0^{\pi} F(x) \cos nx dx.$$

On integrating by parts and making use of (7) and (9), we find

$$A_n = -\frac{2}{\pi n}\int_0^{\pi} \frac{f(x) + f(\pi - x)}{\pi} \sin nx dx$$

$$= -\frac{1}{\pi n}\left[b_n - (-1)^n b_n\right],$$

$$A_n = \begin{cases} 0 & \text{when } n \text{ is even,} \\ -\dfrac{2}{\pi n} b_n & \text{when } n \text{ is odd;} \end{cases}$$

$$\tfrac{1}{2}A_0 = \frac{1}{\pi}\int_0^{\pi} F(x)dx = \frac{1}{\pi}\left[\pi F(\pi) - \frac{1}{\pi}\int_0^{\pi} xf(x)dx - \frac{1}{\pi}\int_0^{\pi} xf(\pi - x)dx\right],$$

or upon putting $u = \pi - x$ in the last integral, $A_0 = 0$. This, of course, is to be expected since, from (6), $F(\frac{1}{2}\pi) = 0$. Thus we have

$$F(x) \sim -\frac{2}{\pi} \sum_{k=0}^{\infty} \frac{b_{2k+1}}{2k+1} \cos (2k+1)x.$$

In this case the even harmonics are absent and the odd harmonics are again reduced by the factor $1/(2k+1)$.

Lastly in the important case in which $f(x)$ satisfies both conditions (I) and (II), we see that

$$f(x) \sim \sum_{k=0}^{\infty} b_{2k+1} \sin (2k+1)x$$

and that $F(x)$ has again the expansion just given.

As an example let

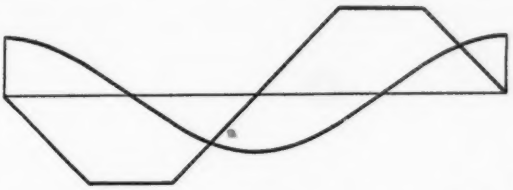
$$f(x) = \begin{cases} x & \text{from } x = 0 \text{ to } x = \frac{1}{3}\pi, \\ \frac{1}{3}\pi & \text{“ } x = \frac{1}{3}\pi \text{ “ } x = \frac{2}{3}\pi, \\ \pi - x & \text{“ } x = \frac{2}{3}\pi \text{ “ } x = \pi. \end{cases}$$

Then

$$f(x) = 2\pi^{-1}\sqrt{3}\left(\sin x - \frac{\sin 5x}{5^2} + \frac{\sin 7x}{7^2} - \frac{\sin 11x}{11^2} + \cdots\right),$$

$$F(x) = -4\pi^{-2}\sqrt{3}\left(\cos x - \frac{\cos 5x}{5^3} + \frac{\cos 7x}{7^3} - \frac{\cos 11x}{11^3} + \cdots\right).$$

The excellent “cosine wave” shown in the figure is made up of parabolas and



straight lines. The graphical method noted above led to the belief that it was probably an actual cosine wave.

ALGEBRAIC FIRST INTEGRALS OF ALGEBRAIC DIFFERENTIAL EQUATIONS

By Solomon Hurwitz, New York, N. Y.

It was shown by Abel that if y is an algebraic function of x and if $u = \int y\,dx$ is also an algebraic function of x, then u is a rational function of x and y. Fuchs[1] stated an analogous theorem for differential equations of the first order: If $F(x, y, y') = 0$, where F is a polynomial, has for a general integral $V(x, y) = c$, V being an algebraic function of x and y, then there exists a general solution $R(x, y, u) = c$, where $u(x,y) = y'$ is the algebraic function defined by our differential equation, and R is a rational function of x, y and u. In this note we prove a theorem for differential equations of order n which yields Fuchs' theorem as an immediate corollary. Our method of proof is different from that of Fuchs.

Theorem: *If* $F(y^{(n)}, y^{(n-1)}, \cdots, y', y, x) = 0$, *where* F *is a polynomial, has an algebraic first integral* $V(y^{(n-1)}, \cdots, y', y, x) = c$, V *being an algebraic function in its arguments, then there exists a first integral* $R(y^{(n-1)}, \cdots, y', y, x, u) = C$,

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[1] Sitzungsberichte der Berliner Akademie, vol. 2 (1884), pp. 1171–1177.

where $u(y^{(n-1)}, \cdots, y', y, x) = y^{(n)}$ *is the algebraic function defined by our differential equation and* R *is a rational function in its arguments.*

I: The necessary and sufficient condition that $V = c$ be a first integral is that

$$\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}y' + \frac{\partial V}{\partial y'}y'' + \cdots + \frac{\partial V}{\partial y^{(n-1)}}u = 0. \tag{1}$$

For, we have

$$\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}y' + \frac{\partial V}{\partial y'}y'' + \cdots + \frac{\partial V}{\partial y^{(n-1)}}y^{(n)} = 0.$$

Hence if $y^{(n)} = u$, it follows that (1) must be satisfied. The condition is necessary. If (1) is satisfied it follows that $y^{(n)} = u$. The condition is therefore sufficient.

II. By hypothesis there exist equations of the form:

$$V^m + \beta_1(y^{(n-1)}, \cdots, y', y, x, u)V^{m-1} + \beta_2 V^{m-2} + \cdots + \beta_m = 0, \tag{2}$$

where the β's are rational in $y^{(n-1)}, \cdots y', y, x$ and u. Let

$$V^p + \alpha_1(y^{(n-1)}, \cdots, y', y, x, u)V^{p-1} + \alpha_2 V^{p-2} + \cdots + \alpha_p = 0$$

be an equation of the class (3) which is of minimum degree in V. By differentiating we get:

$$V^{p-1}\left[p\frac{\partial V}{\partial x} + \frac{\partial \alpha_1}{\partial x}\right] + V^{p-2}\left[(p-1)\alpha_1\frac{\partial V}{\partial x} + \frac{\partial \alpha_2}{\partial x}\right] + \cdots + \left[\alpha_{p-1}\frac{\partial V}{\partial x} + \frac{\partial \alpha_p}{\partial x}\right] = 0,$$

$$V^{p-1}\left[p\frac{\partial V}{\partial y} + \frac{\partial \alpha_1}{\partial y}\right] + V^{p-2}\left[(p-1)\alpha_1\frac{\partial V}{\partial y} + \frac{\partial \alpha_2}{\partial y}\right] + \cdots + \left[\alpha_{p-1}\frac{\partial V}{\partial y} + \frac{\partial \alpha_p}{\partial y}\right] = 0,$$

. . . . . .

$$V^{p-1}\left[p\frac{\partial V}{\partial y^{(n-1)}} + \frac{\partial \alpha_1}{\partial y^{(n-1)}}\right] + V^{p-2}\left[(p-1)\alpha_1\frac{\partial V}{\partial y^{(n-1)}} + \frac{\partial \alpha_2}{\partial y^{(n-1)}}\right]$$
$$+ \cdots + \left[\alpha_{p-1}\frac{\partial V}{\partial y^{(n-1)}} + \frac{\partial \alpha_p}{\partial y^{(n-1)}}\right] = 0.$$

Multiplying these equations by $1, y', \cdots, y^{(n-1)}, u$, respectively and adding, we get

$$V^{p-1}\left[\frac{\partial \alpha_1}{\partial x} + \frac{\partial \alpha_1}{\partial y}y' + \cdots + \frac{\partial \alpha_1}{\partial y^{(n-1)}}u\right] + V^{p-2}(A_2) + \cdots + (A_p) = 0. \tag{3}$$

The coefficients of the V's in this equation are rational in $y^{(n-1)}, \cdots y', y, x$, and u; for, if $\alpha_i = R_i(y^{(n-1)}, \cdots y', y, x, u)$,

$$\frac{\partial \alpha_i}{\partial y^{(k)}} = \frac{\partial R_i}{\partial y^{(k)}} + \frac{\partial R_i}{\partial u}\frac{\partial u}{\partial y^{(k)}}.$$

The first two partial derivatives on the right are obviously rational in $y^{(n-1)}, \cdots, y', y, x$, and u, and by means of the given differential equation we can express $\partial u/\partial y^{(k)}$ rationally in the same quantities. Hence (3) belongs to the class (2) and is of degree $p-1$ in V. Its coefficients must therefore vanish identically. That is

$$\frac{\partial \alpha_1}{\partial x} + \frac{\partial \alpha_1}{\partial y}y' + \cdots + \frac{\partial \alpha_1}{\partial y^{(n-1)}}u = 0.$$

By (I), $\alpha_1 = R(y^{(n-1)}, \cdots, y', y, x, u) = C$ is a first integral of our differential equation.

QUESTIONS AND DISCUSSIONS

EDITED by R. E. GILMAN, Brown University, Providence, Rhode Island.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A PROPOSED NOTATION FOR EXPONENTIALS

By WILLIAM R. RANSOM, Tufts College

The notation "exp x" to represent "the natural base, $e = 2.718+$, with the exponent x," has found some favor because of the ease with which it can be printed or typewritten. Writers who need a greater variety of exponentials may welcome a more complete notation in which the base may be indicated. Suppose for example that

$$(B\}\exp = x)$$

were to find favor for representing "the base, B, with the exponent, x." This permits any degree of complication in either base or exponent, makes it unnecessary to distinguish in size or level between base and exponent, and creates no confusion with existing notations

The symbol "}" is easily managed on any typewriter and is easy for a printer[1] to procure. It may be thought of as locking a parenthesized base to a parenthesized exponent, while the ordinary parentheses, which embrace the whole symbol, have their usual function of indicating that a single quantity is denoted by the array. The symbol "}" indicates and separates base and exponent, just as the solidus, "/", indicates and separates numerator and denominator.

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[1] The Banta Publishing Co., of Menasha, Wis., which prints this Monthly, has the symbol in stock for use on a monotype machine.

When the base is $e=2.718+$, the notation $(e\{\exp=x)$ may condense to the established form "exp x," just as the symbol "$\log_n x$" condenses to the familiar "$ln\ x$" when $n=e$.

Editorial Comment on Professor Ransom's Paper.

Professor Ransom suggests a new symbol for exponentials with a general base which, it will be granted, is distinctive. It cannot be so readily granted that it has the merit of brevity. Moreover, unlike its prototype "exp x," it cannot be immediately set up on a linotype.

QUESTION No. 58: Does the exponential with general base occur frequently enough in mathematics to warrant the general adoption of a special new notation for it, and if so, is Professor Ransom's suggested symbol likely to meet with sufficient approval to be generally adopted?

R. E. G.

THE ARC OF THE ELLIPSE

By ROGER A. JOHNSON, Hunter College

An error in the new Encyclopaedia Britannica leads to the consideration of some expressions for the approximate length of arc of the ellipse which are indeed to be found in the literature,[1] but which are perhaps not so well known to all readers of the MONTHLY as they deserve.

In the article, "Ellipse," in the Encyclopaedia, occurs the following sentence: "Kepler's approximation $\pi(a+b)$ [to the length of an ellipse whose semi-axes are a and b] is about 1/200 too large, and $\pi\sqrt{(a^2+b^2)}$ is about as much too small; their mean errs by about 1/3000."

There is obviously an error in the second expression given; we see below that a factor 2 should appear under the radical sign. Moreover, as to each approximation, it does not seem plausible that the error should be the same, independent of the eccentricity of the ellipse. Further consideration shows that apart from the typographical error just indicated, the Encyclopaedia statement is erroneous in several respects.

If the semi-axes of an ellipse are a and b, its eccentricity is $e=c/a$, where $c^2=a^2-b^2$. Then it is well known that the length of a quadrant of an ellipse is

$$s = a\int_0^{\pi/2}\sqrt{1-e^2\sin^2\phi}\,d\phi = \frac{1}{2}\pi a\left[1-\frac{1}{4}e^2-\frac{3}{64}e^4-\frac{5}{256}e^6-\frac{175}{16384}e^8\cdots\right];$$

and values of this function have been tabulated (as for instance in Peirce's *Short Table of Integrals*, page 121). Now let R represent the radius of a circle whose circumference equals that of the ellipse, so that R/a is given by the expression in brackets. Now it is to be shown that good approximations for R are (1) the arithmetic mean of a and b; (2) the square root of their mean square;

[1] See, for instance, Appell, *Eléments d'analyse Mathématique*, 1905, page 186, where the approximations herein designated as R_1 and R_3 are to be found.

(3) their geometric mean. A truly remarkable approximation is the mean of (1) and (2).

Expressing each of these in terms of a and e, and expanding each by the binomial theorem, we have:

$$R_1 = \frac{a+b}{2} = a\left[1 - \frac{1}{4}e^2 - \frac{4}{64}e^4 - \frac{8}{256}e^6 - \frac{320}{16384}e^8 \cdots\right],$$

$$R_2 = \left(\frac{a^2+b^2}{2}\right)^{1/2} = a\left[1 - \frac{1}{4}e^2 - \frac{2}{64}e^4 - \frac{2}{256}e^6 - \frac{40}{16384}e^8 \cdots\right],$$

$$R_3 = (ab)^{1/2} = a\left[1 - \frac{1}{4}e^2 - \frac{6}{64}e^4 - \frac{14}{256}e^6 - \frac{616}{16384}e^8 \cdots\right]$$

$$\frac{1}{2}(R_1 + R_2) = a\left[1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 - \frac{5}{256}e^6 - \frac{180}{16384}e^8 \cdots\right].$$

It follows then that Kepler's approximation R_1 is too small, and that R_2 is too large, the error in each case being of the order of $e^4/64$. The mean of R_1 and R_2 is too small by approximately $5e^8/16384$.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College of the City of New York.

All books for review should be sent directly to the editor of this department and not to any of the other editors or officers of the Association.

The Rhind Mathematical Papyrus, British Museum 10057 and 10058. Vol. I. Free Translation and Commentary, by A. B. Chace, Chancellor, Brown University, with the assistance of H. P. Manning, Associate Professor of Mathematics, Brown University, Retired. Bibliography of Egyptian Mathematics, by R. C. Archibald, Professor of Mathematics, Brown University. Mathematical Association of America, Oberlin, Ohio, U. S. A., 1927. Pages 1–210. Volume II, Photographs, Transcription, Transliteration, Literal Translation, by A. B. Chace · · · L. Bull, Associate Curator in the Egyptian Department, Metropolitan Museum, New York, H. P. Manning, · · · Bibliography of Egyptian and Babylonian Mathematics (Supplement), by R. C. Archibald · · · . The Mathematical Leather Roll in the British Museum, by S. R. K. Glanville, Department of Egyptian and Assyrian Antiquities, British Museum, · · · 1929.

The publication of Chace's edition of the Rhind papyrus is an event of importance to all mathematicians interested in the history of their science. It marks the culmination of the efforts of many students of Egyptian mathematics, endeavoring to present to the modern reader the complete and exact form of the ancient papyrus, to supply an exact translation of it and to give a philosophic

interpretation of Egyptian processes in mathematics. The Chace edition represents all that painstaking study and liberality in financial expenditure could accomplish at the present time.

Before the discovery of the Rhind papyrus, little was known of Egyptian mathematics, except the fragmentary statements of Greek writers. Thales and Pythagoras had travelled in Egypt and acquired Egyptian lore. But the Egyptian mathematics which percolated into Greek books which have come down to the present time seem unworthy of a people which had erected the Egyptian pyramids. The Rhind papyrus was found about 1858, in a building near the great monument Ramesseum, at Thebes. It came into the possession of A. Henry Rhind from whom its name is derived. In 1864 it was purchased by the trustees of the British Museum. The papyrus roll was broken in some places and parts were missing. The trustees of the British Museum, in 1869, authorized the preparation of a facsimile edition and a descriptive text of the papyrus. Samuel Birch, Keeper of the Egyptian antiquities of this Museum, prepared plates, but the progress of the work was delayed. In the Spring of 1872, the archeologist August Eisenlohr of Heidelberg visited England and secured from Birch proofs taken from these plates. After five years of study, in which he was assisted by Moritz Cantor, the historian of mathematics, Eisenlohr brought out his edition of the Rhind papyrus in 1877. He did this, however, without securing the consent of the trustees of the British Museum to use Birch's plates. Besides the reproduction of the papyrus itself, Eisenlohr gave a commentary, as well as a translation of the papyrus into German and a transcription of the hieratic script in which the papyrus is written into the more picturesque hieroglyphic script. During the half century following, Eisenlohr's very creditable publication was the only source from which mathematicians could obtain a knowledge of the contents of the Rhind papyrus. Meanwhile, in 1898, the British Museum issued a beautiful lithographic "facsimile" of the papyrus. By "facsimile" we are not to understand an exact reproduction by the aid of photography; it was simply an imperfect hand copy, containing many little deviations from the original. As Eisenlohr used the British Museum plates, his representation of the papyrus was no better; besides he made some slight alterations of his own.

During the fifty years following 1877, great strides were made in the mastery of the languages of ancient Egypt. It became evident that a new translation and commentary of the Rhind papyrus was desirable. In 1923 this was accomplished in a masterly way by T. Eric Peet, professor of Egyptology in the University of Liverpool. By a strange good fortune some of the missing parts of the papyrus were found in 1922, in the possession of the New York Historical Society. Copies of these fragments were prepared, and Peet succeeded in finding the proper places in the papyrus for 25 of the 40 fragments. Using the papyrus itself, not the "facsimile" of it, Peet gave a hieroglyphic transcription and a translation into English. Critics found this a very able piece of work, notwithstanding occasional slips of perhaps minor importance.

For some years preceding the appearance of Peet's translation and commentary, Arnold B. Chace, Chancellor of Brown University, had made an intensive study of the Rhind papyrus. In some details his conclusions were not in entire agreement with those of Peet. Chace and his co-workers felt that there was still room for a new edition of the papyrus, especially as no real facsimile of the papyrus, and of the fragments possessed by the New York Historical Society, were available to the general reader. As the outcome of Chace's enthusiasm and liberality, we have the present magnificent edition, of which the second volume contains the photographic reproduction of the papyrus as it appears at the present time in the British Museum, and of the New York fragments. There is a satisfaction in examining an exact copy. When told that a pair of legs walking forward represents "addition," and when walking backward represents "subtraction," we like to see how much of this statement is due to the imagination of the modern investigator, and how much to the graphic technique of the ancient scribe. Our inspection leads to the conclusion that the modern imaginative faculty operated in full force.

The different parts of the papyrus are shown by 31 photographs. In making the hieroglyphic transcription, the transliteration, and literal translation, 109 plates are used. This is not the place to enumerate the many minute new interpretations. Suffice it to say that a dozen small fragments not located by Peet, are in the Chace edition assigned to what seem to be their proper places in the papyrus.

As indicated on the title page Chancellor Chace had as collaborators, H. P. Manning and R. C. Archibald, both of Brown University, and Ludlow Bull of the Egyptian department of the Metropolitan Museum in New York. Mr. S. R. K. Glanville of the British Museum supplied a brief account of a mathematical leather roll in that Museum. It is an Egyptian document which appears to be of greater significance than at first thought.

A valuable feature of the Chace edition is the bibliography of Egyptian mathematics prepared by Archibald. The many informative and critical remarks are particularly valuable to readers who have not the time or opportunity to consult the original sources. The very last entry is a pre-publication notice of V. V. Struve's account of the Golenishchev mathematical papyrus of the Moscow Museum of Fine Arts, which will appear in *Quellen und Studien zur Geschichte der Mathematik, Abteilung A: Quellen*, Berlin, 1930. Parts of this Egyptian papyrus have been deciphered and published before. It contains startling revelations on early Egyptian mathematics, and may rival in importance the Rhind papyrus.

FLORIAN CAJORI

Über die Verhältniszahl des Goldenen Schnitts, die Reihe der mit ihr zusammenhängenden ganzen Zahlen und eine aus dieser abgeleitete Reihe. By Ludwig Kaiser. Leipzig, B. G. Teubner. Paper, viii+124 pages. Price: RM 7.50.

If the line-segment AB be divided internally at C in what Euclid called

extreme and mean ratio, *i.e.* so that $(AB/AC)=(AC/CB)$, it is easy to show that the value of each of these ratios is the positive root of $q^2=q+1$. The author gives nearly forty pages to a discussion of the arithmetic of the number q so defined. About its outcroppings in geometry, phyllotaxis, aesthetics, and elsewhere he has nothing to say.[1] The treatment is elementary and not hurried; theorems are found by exploration, illustrated in numbers, and proved usually by complete induction. It is shown that any integral power of q can be expressed in the form $aq+b$ where a and b are consecutive terms of the doubly endless Fibonacci series

$$g_1 = g_2 = 1, \quad g_n = g_{n-1} + g_{n-2}, \quad (n = \cdots, -2, -1, 0, 1, 2, 3, \cdots).$$

The derived series mentioned in the title is

$$g_1' = 1, g_2' = 3, \quad g_n' = g_{n-1}' + g_{n-2}' = g_{n-1} + g_{n+1} = (g_{2n}/g_n).$$

The author uses the word "Kepler" where we have the habit of saying "Fibonacci" or "Pisano," uses g to remind us of golden and, if he is aware that these are the Lucas u and v series concerning which Dickson gives more than a hundred references in chapter XVII of volume I of his *History of the Theory of Numbers*, he does not anywhere show it.

Because the positive integral powers of q can be expressed as periodic continued fractions of the form $(c; d, d, d, \cdots)$, the properties of continued fractions are worked out in detail and, since c and d are found always to belong to one or the other of the above series, the illustration of these properties leads to many interesting theorems.

The remainder of the book is devoted to a study of the Fibonacci numbers but no chance is missed to establish analogous relations for the terms of the associated series wherever possible. Among the topics are: general term, sum, periodicity quâ any integral modulus, types of period, appearance and repetition of primes, and the so-called Pell equation $x^2-5y^2=1$. The author does not make the claim that any of his results are new. Appended tables contain the factored forms of g_n and g_n' $(n=1, 2, \cdots, 40)$ and a summary of the characteristics of the remainder-periods for all prime moduli up to 41 and for about twenty other primes.

The book can be read with profit by any student who knows the elements of the theory of numbers. The style is pleasant; the author, like a good teacher, feels free to pause for the examination of any sort of interesting by-product but when a proof is begun it is carried directly forward and the conclusion is nailed down w. z. b. w. The headings of each of the twenty-five chapters are as ample as the title of the book itself and make an index unnecessary. The

[1] See this MONTHLY, vol. 25 (1918), pp. 232–8, where Professor R. C. Archibald gives an extensive bibliography of such applications. Incidentally he shows that the glamorous words "golden section" are less than a hundred years old. See also Tropfke, *Geschichte der Elementar-Mathematik*, vol. IV, p. 187.

appearance of the page and the printing and proof-reading are of that degree of excellence which we have learned to expect from the house of Teubner.

NORMAN ANNING

Premières Leçons de Géométrie Analytique et de Géométrie Vectorielle à l'usage des élèves de la classe de mathématiques et des candidats aux grandes écoles. By E. Lainé. Librairie Vuibert, Paris, 1929. 47 pages.

Recently there has been a tendency in the classical preparation leading up to the university courses in France to introduce vector notation in analytic geometry, kinematics, rational mechanics and differential geometry along with the usual analytic notation, such as one finds, for instance, in Appell's *Traité de Mécanique Rationnelle.* In these introductory courses the theory of systems of vectors is treated quite fully. In the present book just enough of vector analysis is given so that the average student can get a glimpse of some of its possibilities. It gives the usual discussion of vectors: vector addition, scalar multiplication, scalar product, vector product, and derivative of a vector. It applies the elementary theory to the study of straight lines, circles, planes, circular motion and systems of vectors. The theory is followed by a set of 23 problems, several of which are taken from *baccalauréat* examination papers, designed for students who have a knowledge of elementary calculus.

There are no important typographical errors. On page 28, line 12, x^0 should read x_0. On page 33, figure 19, the letters G and G_1 are omitted. The derivation of the expressions for the components of the vector product of two vectors (page 17) might have been simplified.

The geometric and vector system applications might well find their place in American elementary analytic geometry and mechanics texts, since vector analysis, dealing directly with geometric objects instead of indirectly by means of their components, gives a greater simplicity and elegance of presentation in these subjects. A more advanced text along the same line, as clearly written as this one, would be extremely useful to both students and instructors.

W. E. BYRNE

A History of Mathematical Notations. Volume II. Notations Mainly in Higher Mathematics. By Florian Cajori, Ph.D. Chicago, The Open Court Publishing Company, 1929. xvii+367 pages. $6.00.

This work is the second volume of Professor Cajori's *History of Mathematical Notations.* As the author indicates in his introduction, "The task of making a complete collection of signs occurring in mathematical writings from antiquity down to the present time transcends the endurance of a single investigator." Indeed, the invention of new symbols is as confusing as the change in the styles of women's clothes. After the mathematical world had been educated to the meaning and value of symbols, many an author exercised his ingenuity on the making of them. Surely the study of such a work as this reveals the ill effects

of this multiplicity of signs and should "afford a more intense conviction of some form of organized effort to secure uniformity."

As has been stated in a review of the first volume of this work, this *History of Mathematical Notations* provides an inexhaustible mine of facts and bibliography for the historian of mathematics and for the teacher of its history. The painstaking, careful and exhaustive labor extending over many years and resulting in a work of such magnitude places the mathematical-historical world in a position of deep indebtedness to the author. If all the uses to which these are put could be gathered together and presented in one continuous scroll, appreciation due his results might be adequately shown. Perhaps a greater reward would lie in his own expression "that this *History* will not have been written in vain" if advance in the spread of new results in mathematics can be brought about through international action on adoption of symbols.

The symbols associated with almost any topic in mathematics will be found in one or the other volumes of this work. Among these, none are more fully and convincingly presented than the symbols of the Differential and Integral Calculus. "The history of the growth of the calculus notations is not only interesting, but it may serve as a guide in the invention of fresh notations in the future" and here are included symbols for fluxions, differentials and derivatives, partial differentials and partial derivatives with the names of many prominent mathematicians beginning with Newton and Leibniz.

The author presents an intensive study entitled: "Survey of Mathematical Symbols used by Leibniz" because of the "leading rôle played by Leibniz in the development of notations" and because "no mathematician has seen more clearly than Leibniz the importance of good notations in mathematics." (pp. 180–1). This article makes fascinating reading and is a relief from the succession of symbols whose multiplicity makes one's head fairly reel, although this amount of detail is of incalculable value to the student of a symbol. Following this special study are nine pages of tables of symbols used by Leibniz in his manuscripts and in the papers he published.

Many interesting phases of the invention of symbols are found. There is, for instance, the casual appearance of symbols now well-known as in the case of π. There is the adoption of a symbol already used by a writer through the influence of a great mathematician as in the case of i. There is the credit accorded to a man as the sole originator of a symbol when other men had anticipated him as in the case of the notation for partial derivatives now in use. We find the appearance of a symbol for a concept that had been long present in the writings of mathematicians, such as the familiar symbol "to signify infinite number." Our attention is drawn to the fact that: "In the nineteenth century the need of more compact notations asserted itself" and there resulted such representations as Salmon's $(a_1, b_2, c_3 \cdots)$. Exception is taken to the extremes to which certain leaders in mathematical thinking are going in the use of symbols in the phrase "notations for matters that can more conveniently be expressed by ordinary words or in less specialized symbolism." (p. 77).

The reviewer must find some matters from which to dissent. Attention has

been called in an article appearing in School Science and Mathematics, vol. 24 (1924), p. 509, to an instance of reading into the works of early mathematicians modern concepts. This is here illustrated by the statements that "Euclid in his *Elements* represented general number by segments of lines" and that "In Leonardo of Pisa's *Liber Abaci* (1202), the general representation of given numbers by small letters is not uncommon." (p. 1.) Is not this unjustifiable enlargement of the field of symbolic representation! Jordanus Nemorarius did use letters to represent numbers quite as they are used at present, and he belongs in a different category from the earlier mentioned authors.

Some fifteen pages are devoted to "The Evolution of the Dollar Mark." It may be hypercritical to question the presence of this symbol in the work although the title leaves a loophole for its entrance. To have the author's views and evidence so completely presented in permanent form is an asset to a library, however. Such a widely divergent theory, backed by a mass of manuscript evidence, is suggested by an equally eminent authority in this field that a complete presentation of the latter is highly desirable. The two positions should be weighed over against each other.

The connected account is interrupted at times by matter (p. 239) which makes the work seem more encyclopedic in character, for the sporadic use of certain symbols by an erratic writer has no bearing on the general problem.

Considerable space toward the end is devoted to lessons to be learned and to the solution of the whole problem. Quotations from many writers are given. A fine section entitled: "Empirical Generalizations on the Growth of Mathematical Notations" shows the author's masterly and scholarly grasp of the problem that he presents. The conclusion of the whole matter as its beginning is: "Agreements to be reached by International Committees the only Hope for Uniformity in Notations." It cannot be overemphasized that: "In the endeavor to secure the universal adoption of mathematical symbolism international cooperation is a *sine qua non*." Moreover, "no attempt should be made to set up at any one time a full system of notation for any department of mathematics." (p. 349). The last words in the book are too significant to omit: "· · · when in their publications, mathematicians, by a gentlemen's agreement, shall abide by the decisions of such committees." (p. 350).

The contents are: Topical Survey of Symbols in Arithmetic and Algebra (Advanced Part); Symbols in Modern Analysis; Symbols in Geometry (Advanced Part); The Teachings of History; with many subheadings. Twenty illustrations enrich the subject matter.

The strange symbols which pepper the pages of this work indicate the immense amount of labor and expense which went into the printing of it. The unique contribution to mathematical learning which The Open Court Publishing Company makes through its publications is again strikingly illustrated in this one. These are needed and only through such a high-minded agency can the need be met.

LAO G. SIMONS

THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teacher's agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to department or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3419. *Proposed by the late F. P. Matz.*

Determine the law of force in order that the orbit be a Cassinian oval.

3420. *Proposed by the late G. B. M. Zerr.*

Let $x=f(t)$ be the equation giving the horizontal distance of a projectile in a resisting medium in terms of the time t. Prove that the vertical distance is given by

$$y = -gf(t)\int \frac{dt}{f'(t)} + g\int \frac{f(t)}{f'(t)}dt + Af(t) + B,$$

where A and B are constants.

3421. *Proposed by Otto Dunkel, Washington University.*

A convex polygon of n sides may be divided into triangles by its diagonals which intersect only at their extremities. Derive an expression for the number of ways in which this may be done.

3422. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Prove that the polar reciprocal of a sphere with respect to a given sphere is a quadric of revolution. Discuss the nature of this quadric.

3423. *Proposed by Morgan Ward, California Institute of Technology.*

Let A and B be two permutable elements of an abstract group, of orders a and b respectively, and let c be the order of $C=AB=BA$. Show that m' divides c, and c divides m, where m is the L.C.M. of a and b, and m' is the product of all of those prime factors of m which appear in a and b raised to different powers, every such factor being raised to the power to which it appears in m.

3424. *Proposed by W. J. Greenstreet, Editor of the Mathematical Gazette.*

Two straight rods OA, OB in the vertical plane make equal angles, α, with the vertical drawn up from the point O. A third rod, PQ, of fixed length l with rings at its ends slides on the two rods. Show that the third rod may assume a position of equilibrium if its centre of mass lies within a portion of the rod PQ of length $l \sin 2u \operatorname{cosec} 2\alpha$, where u is the angle of friction for rings and rods.

UNSOLVED PROBLEMS

2854 [1920, 377]. *Proposed by C. N. Mills.*

Solve the simultaneous equations for x and y

$$x^n + y^n = a_n, \qquad x^{n-1} + y^{n-1} = a_{n-1}.$$

2856 [1920, 377]. *Proposed by O. S. Adams, U. S. Coast and Geodetic Survey.*

Show that for the real domain defined by $+1>x>-1$, and s a positive integer,

$$\frac{1}{(1-x^s)^{1/s}} \int_0^x \frac{dx}{(1-x^s)^{(s-1)/s}} = x + \sum_{n=1}^{\infty} \frac{2(s+2)(2s+2)\cdots(ns-s+2)}{(s+1)(2s+1)\cdots(ns+1)} x^{ns+1},$$

and

$$\frac{1}{(1-x^s)^{(s-1)/s}} \int_0^x \frac{dx}{(1-x^s)^{1/s}} = x + \sum_{n=1}^{\infty} \frac{n!s^n}{(s+1)(2s+1)\cdots(ns+1)} x^{ns+1}.$$

SOLUTIONS

3309 [1928, 93]. *Proposed by Otto Dunkel, Washington University.*

If (1, 2, 3, 4, 5, 6) denotes the determinant of the 6th order whose ith row is

$$x_i^2,\ x_iy_i,\ y_i^2,\ x_i,\ y_i,\ 1, \text{ and if } (i,j,k) = \begin{vmatrix} x_i, & y_i, & 1 \\ x_j, & y_j, & 1 \\ x_k, & y_k, & 1 \end{vmatrix},$$

show that

$$(1,2,3,4,5,6) = (6,1,2)(2,3,4)(4,5,6)(1,3,5) - (1,2,3)(3,4,5)(5,6,1)(2,4,6)$$

is an identity in the 12 independent variables x_i, y_i.

Solution by the Proposer

Denote by ϕ the polynomial (1,2,3,4,5,6) and by ψ the other polynomial on the right. If the set (x_1, y_1) is replaced by (x_2, y_2), (x_3, y_3), (x_5, y_5), (x_6, y_6), ψ vanishes since each of its two parts vanish. If (x_1, y_1) is replaced by (x_4, y_4) the two parts of ψ are equal and cancel and ψ again vanishes. If a cyclic interchange is made in the subscripts of ψ so that 1 2 3 4 5 6 is replaced by 2 3 4 5 6 1 then ψ merely changes sign. Hence, when the subscript 2 is replaced in turn by each of the remaining five subscripts, ψ vanishes, etc. Thus ψ vanishes when the subscript i is replaced by a different subscript j. This is obviously true of ϕ.

The polynomial ϕ has only one term of the form $x_1^2 x_2 y_2 y_3^2 x_4 y_5$ and this comes from the principal diagonal. We shall examine ψ for the presence of this term, considering the two parts separately. There are four determinant factors in each part, and, since the total degree of the term considered is eight, each factor must yield a term of the second degree for the product. Marking the letters x_3, x_5, x_6, y_1, y_4, y_6 which must be rejected, we see that (6 1 2) yields one and only one term which may be used, $x_1 y_2$; similarly (456) yields $x_4 y_5$; then in (135) we must reject y_5 since we already have it, and there is left $x_1 y_3$; and finally in (234) we must reject every term but $x_2 y_3$. Thus the first part yields this term only once with the coefficient unity. In the second part (561) yields only $x_1 y_5$. Then (345) yields only $-x_4 y_3$, since we already have y_5. In (246) the only term which may be used is $-x_4 y_2$; but this would give in the product x_4^2. Hence the second part does not contain the term in question, and it follows that ϕ and ψ have this same term with the coefficient unity. Also neither ϕ nor ψ can be identically zero as polynomials in the twelve independent variables which they contain.

Consider the different determinants $(i\ j\ k)$ by taking all possible combinations of i, j, k so that no two subscripts are the same in the same determinant. Denote the product of all these determinants by π. No single factor of π is identically zero and hence π is not identically zero; also the product $\phi\pi$ is not identically zero, and we can find a set of particular values for the twelve variables such that $\phi\pi$ is not zero for these values. Since $\phi\pi$ is continuous in the twelve variables we can determine a region R for these variables such that $\phi\pi$ does not vanish for any values of the variables selected from this region R. Select a definite set of values from R for all the variables except x_1, y_1. Then for these special fixed values ϕ is a polynomial of the second degree in x_1, y_1 not identically zero. We shall show that it is irreducible. When x_1, y_1 assume the values already selected $x_2, y_2; x_3, y_3; \cdots; x_6, y_6$, ϕ vanishes. Hence, if ϕ consists of two linear factors in x_1, y_1, one factor at least must vanish for three of these sets of points, say $x_2, y_2; x_3, y_3; x_4, y_4$; and then (234) would be zero. But this is impossible since these values are taken from R. Hence ϕ is irreducible. Consider ψ as a polynomial in x_1, y_1 with the same constant values for the other variables that appear in ϕ. The two polynomials are of the second degree in x_1, y_1; one of them is irreducible; and each vanishes for five distinct sets of

values, the values of the constants. Hence we must have $\phi \equiv c_1\psi$, where c_1 is independent of x_1, y_1, and ϕ and ψ are regarded as polynomials in x_1, y_1.

Now regard x_2, y_2 as variables in the two polynomials while the remaining variables assume fixed values from the region R. Then as before we conclude that $\phi \equiv c_2\psi$, where c_2 is independent of x_2y_2. If in the two equations $\phi \equiv c_1\psi$, $\phi \equiv c_2\psi$ we suppose that the twelve variables have the same values which are taken from R, then $(c_2 - c_1)\psi = 0$. But since ϕ does not vanish for these values neither does ψ, and hence $c_2 = c_1$. This shows that neither c_1 nor c_2 depends upon (x_1, y_1) or (x_2, y_2), at least for the region R. Reasoning in the same way we show that $c_1 = c_2 = c_3$ and each is independent of the variables with the subscripts 1, 2, 3; and so on until finally we have $\phi \equiv c\psi$, where c is an absolute constant, at least for R. But since ϕ and ψ have each the same term with the same coefficient unity, we have $c = 1$, and then $\phi \equiv \psi$. Hence the corresponding coefficients in ϕ and ψ are the same for the region R; and we have therefore $\phi \equiv \psi$ without any restriction upon the range of the twelve variables.

As an application of this result suppose that one of the second order determinants is zero, say (123), then the determinant ϕ breaks up into the product of four determinants each of the second order.

Also solved by Paul Wernicke.

3360 [1928, 564]. *Proposed by Bernardo Ig. Baidaff, Buenos Aires.*

Resolve the system $x_i = a^i + bx_{i-1} + b^2x_{i-2} (i = 1, 2, \cdots, n)$, with $x_{-1} = x_0 = 0$. Generalize.

Solution by Paul S. Dwyer, Antioch College.

It is easily found that

$$
\begin{aligned}
x_1 &= a, \\
x_2 &= a^2 + ab, \\
x_3 &= a^3 + a^2b + 2ab^2, \\
x_4 &= a^4 + a^3b + 2a^2b^2 + 3ab^3, \\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{1}
$$

The above results when continued suggest

$$
x_i = \sum_{j=0}^{i-1} k_j a^{i-j} b^j, \qquad i = 1, 2, 3, \cdots, \tag{2}
$$

where $k_0 = k_1 = 1$ and $k_j = k_{j-1} + k_{j-2}$, $j > 1$. The above results in (1) are given by (2), and we shall now show that (2) gives the solution of the system of equations in the problem. Inserting the values of x_i, x_{i-1}, x_{i-2} in the given equation, we shall show that (2) is identically equal to

$$
a^i + b \sum_{j=0}^{i-2} k_j a^{i-j-1} b^j + b^2 \sum_{j=0}^{i-3} k_j a^{i-j-2} b^j. \tag{3}
$$

The first two terms of (2) are $k_0a^i + k_1a^{i-1}b$, while in (3) the corresponding terms

are $a^i+k_0a^{i-1}b$, and these are equal since $k_0=k_1=1$. For any other value of j in (2) we consider $j-1$ and $j-2$ in the first and second sums of (3), and we obtain $(k_{j-1}+k_{j-2})a^{i-j}b^j=k_ja^{i-j}b^j$, $1<j\leqq i-1$. Hence (2) satisfies the given equation.

Note by Otto Dunkel: The numbers k_j are the famous rabbit numbers which form the well known Fibonacci sequence mentioned several times in this Monthly. For example see [1921, 329] the solution of 2809 [1920, 80], where a formula is worked out for such numbers and references to the literature on the subject are given. The formula referred to when simplified reduces to

$$k_j = \frac{1}{2^j}\sum_{p=0}^{m} {}_{j+1}C_{2p+1}5^p,$$

where m is the greatest integer in $\frac{1}{2}j$ and where ${}_{j+1}C_{2p+1}$ is the binomial coefficient.

A generalization is obtained in the equation

$$y_i = c^i + y_{i-1} + y_{i-2} + \cdots + y_{i-m}, \quad m\geqq 1, \quad i = 1,2,3, \cdots,$$

with the initial conditions $y_0=y_{-1}=\cdots=y_{1-m}=0$. This equation goes over into the form of the problem by setting $a=bc$, $x_i=b^iy_i$. The solution of this equation is given by

$$y_i = \sum_{j=0}^{i-1} k_jc^{i-j},$$

where $k_j=k_{j-1}+k_{j-2}+\cdots+k_{j-m}$ with the initial conditions $k_{-1}=k_{-2}=\cdots=k_{1-m}=0$, $k_0=1$. The equation in the k's may be solved by the methods given in the article, *Solutions of a probability difference equation*, in this Monthly, vol. 32 (1925), pp. 354–370. This particular equation is considered on page 365 of that article.

A simpler form of solution may be obtained by using the general methods employed for handling a difference equation with constant coefficients. The y equation may be written

$$f(U)y_i = c^{m+i},$$

where $Uy_i=y_{i+1}$, $U^2y_i=y_{i+2}$, $\cdots$, and where $f(t)=t^m-t^{m-1}-t^{m-2}-\cdots-t-1$. If c is not a root of this equation in t, a particular solution of the difference equation above is given by $f(c)z_i=c^{m+i}$. The homogeneous equation $f(U)y_i=0$ admits the solution

$$-f(c)y_i = s_1k_{i-1} + s_2k_{i-2} + \cdots + s_mk_{i-m},$$

where $s_1, s_2, \cdots, s_m$ are arbitrary constants and k_i is defined as above. Hence the general solution is

$$f(c)y_i = c^{m+i} - [s_1k_{i-1} + s_2k_{i-2} + \cdots + s_mk_{i-m}].$$

The constants s_j are to be determined so that $y_0=y_{-1}=\cdots=y_{1-m}=0$ and it is easily found that $s_j=c^j+c^{j+1}+\cdots+c^m$.

If $f(c)=0, f'(c)\neq 0$ as shown in the article mentioned above [1925, 357]. The particular solution in this case is $f'(c)z_i = ic^{m+i-1}$. The solution of the equation with the given initial conditions is in this case

$$f'(c)y_i = ic^{m+i-1} + s_1' k_{i-1} + s_2' k_{i-2} + \cdots + s_{m-1}' k_{i-m+1},$$

where $s_j' = (m-j)c^{j-1} + (m-j-1)c^j + \cdots + c^{m-2}$.

For the special case of the problem where $m=2$ these solutions assume the simple form

$$(a^2 - ab - b^2)x_i = a^{i+2} - ab^{i+1}k_{i-1} - a^2b^ik_i,$$

if the coefficient of x_i is not zero. If it is zero, then the solution may be written in the form

$$(2a - b)x_i = ia^{i+1} + b^{i+1}k_{i-1}.$$

The second case, in which $f(c)=0$, may be obtained from the first by differentiation with respect to c and then performing certain reductions. It is simpler, however, to proceed as above.

3383 [1929, 338]. *Proposed by S. A. Corey, Des Moines, Iowa.*

A bag contains $(m+n)$ balls, each of which may be either black or white with equal probability, m being the number of balls of one color and n being the number of balls of the other color. A white ball is dropped into the bag, and then a ball is drawn out at random and found to be white. What is now the chance that of the original balls n were white and m black?

Solution by R. E. Moritz, University of Washington.

According to the conditions of the problem the original balls in the bag are either m black and n white, or n black and m white, and the probability of either alternative is exactly one-half. Of a sufficiently large number $(2N)$ of such bags approximately N will contain n white balls and the other N will contain m white balls. Suppose a white ball is dropped into each of the $2N$ bags and a random drawing is made from one of the bags, if the drawing occurs from a bag containing now $n+1$ white balls, the probability of the white ball being drawn is $(n+1)/(m+n+1)$, while if the drawing occurs from a bag containing $m+1$ white balls the probability of the ball being white is $(m+1)/(m+n+1)$.

Take N a multiple of $m+n+1$, say $N=k(m+n+1)$, and suppose that one drawing is made from each of the bags, then the drawings from the bags containing $n+1$ white balls will yield $N(n+1)/(m+n+1)=k(n+1)$ white balls and the drawings from bags containing $m+1$ white balls will yield $N(m+1)/(m+n+1)=k(m+1)$ white balls, in other words, on the long run, out of every $m+n+2$ white balls that are drawn, $n+1$ come from bags which originally contained n white balls, the remaining $m+1$ come from bags which originally contained m white balls.

Hence the probability that a given bag originally contained n white balls is $(n+1)/(m+n+2)$.

Also solved by Andrew G. Clark, Michael Goldberg and Paul Wernicke.

3384 [1929, 338]. *Proposed by Nathan Altshiller-Court. University of Oklahoma.*

A circle (C) touches an equilateral hyperbola (H) in A and passes through the diametric opposite B of A on (H). (1) Prove that the third point common to the two curves is the diametric opposite of A on (C). (2) The lines joining A to the ends of the diameter of (C) perpendicular to AB are parallel to the asymptotes of (H). (3) If a line through A meets the two curves again in P and Q, show that BP, BQ are equally inclined to BA.

I. Solution by A. Pelletier, Montreal, Canada.

Let O and O' be the centers of (H) and (C), respectively; let OM and OM' be the intercepts on the asymptotes of the tangent at A. Draw the diameter AA' of (C), and let its intercepts on the asymptotes be OD and OD'. Since the curve is an hyperbola $MA=AM'$, and since it is equilateral each of these lengths is equal to OA. Then $OO'=DO'=O'D'$; for $\angle OD'O'=\angle AM'O=\angle AOM'=\angle D'OO'$. Hence $AD=AO'-DO'=O'A'-O'D'=D'A'$. This shows that A', the diametric opposite of A on (C), lies also on (H).

Draw EE', the diameter of (C) passing through O. Since $O'E=O'A$ and $O'O=O'D$, the chord EA is parallel to the asymptote OD, and, consequently, AE' is parallel to the other asymptote OD'.

Draw any line through A meeting (C) in P and (H) in Q. Let the intercepts of PQ and BQ on the asymptotes be OK', OK, and OL', OL, respectively. From the properties of the hyperbola it follows that $K'A=QK$, $BL'=LQ$, and the triangle LKQ is isosceles, since KL passes through O. Also $\angle ABP=\angle M'AP=\angle MAQ$. The two triangles MAK and OBL have $\angle MKA=\angle OLB$, and $\angle AMK=\angle BOL$. Hence $\angle ABQ=\angle OBL=\angle MAQ=\angle ABP$. This completes the proof.

II. Solution by Otto Dunkel, Washington University.

Let the tangents to the circle (C) at A and B meet in T. Since AT is also tangent to (H) at A, and since M, the mid-point of AB, is the center of the hyperbola (H), the tangent to (H) at B is parallel to AT. The line through M parallel to AT is the diameter of (H) conjugate to AB, and hence these two diameters are harmonically separated by the asymptotes. But, since the asymptotes are perpendicular to each other, they bisect the angles between the two diameters. Draw the line TM cutting (C) in E and E', then it follows that AE and AE' are parallel to the asymptotes.

The hyperbola (H) is generated by two projective pencils BQ and AQ with centers A and B on (H). Consider the difference of angles $\angle ABQ-\angle BAQ$. When the ray BQ is along the tangent at B this difference is equal to $\angle BAT$; when Q is at A this difference is again equal to $\angle BAT$. When the ray AQ passes through E, $\angle BAQ=90°-\frac{1}{2}\angle BAT$, and $\angle ABQ=90°+\frac{1}{2}\angle BAT$, since AE has the direction of an asymptote. Hence the difference is again $\angle BAT$. This suffices to show that the two pencils at A and B have opposite sense and

are equal. Hence $\angle ABQ - \angle BAQ = \angle BAT$. But since P is on (C) $\angle BPA = \angle BAT$, and then $\angle ABQ = \angle BAQ + \angle BPA = 180° - \angle ABP$. This proves (3), and (1) easily follows. For, when AQ passes through the intersection C of (H) and (C), BP and BQ coincide in BC, and BC is therefore perpendicular to AB. Hence AC is a diameter of (C).

Also solved by J. H. Neelley, Margaret M. Young, and Paul Wernicke.

3385 [1929, 397]. *Proposed by J. Rosenbaum, Milford, Conn.*

In any pentagon, $A_1A_2 \cdots A_5$, P_1, $P_2 \cdots P_5$ are the middle points of the sides A_1A_2, A_2A_3, $\cdots$, A_5A_1. The points M_1, M_2, $\cdots$, M_5 are the midpoints of P_1P_3, P_2P_4, $\cdots$, P_5P_2. Prove that the lines A_5M_1, A_1M_2, $\cdots$, A_4M_5 are concurrent at a point O, such that the vectors OA_1, OA_2, $\cdots$, OA_5 form a closed polygon.

Solution by Otto J. Ramler, The Catholic University of America.

Choose as the origin of vectors the centroid O of the vertices of the pentagon, and let a_i, p_i, m_i represent the vectors from O to A_i, P_i, M_i, respectively. Then $\sum a_i = 0$, and the vectors OA_i form a closed polygon. From the definition of the points we have $2p_i = a_i + a_{i+1}$, $2p_{i+2} = a_{i+2} + a_{i+3}$, $4m_i = (2p_i + p_{i+2})$. Hence $4m_i = a_i + a_{i+1} + a_{i+2} + a_{i+3} = -a_{i-1}$, and this means that $A_{i-1}M_i$ passes through O.

Also solved by J. W. Clawson. Michael Goldberg, W. M. Miller, and A. Pelletier.

3386 [1929, 397]. *Proposed by H. E. Trefethen, Colby College.*

If the incircle passes through the centroid of a triangle, find positive integral values for the sides a, b, c.

Solution by Norman Anning, University of Michigan.

By hammer and tongs or by reference to Hobson's *Plane Trigonometry*, 3rd edition, p. 200, the value of GI, the distance from centroid to incenter, can be found. When it is set equal to the inradius there results

$$(a + b + c)(5a^2 + 5b^2 + 5c^2 - 6bc - 6ca - 6ab) = 0.$$

There are no positive integers which satisfy $a+b+c=0$. If we put $a=cx$, $b=cy$, our problem reduces to that of finding rational points on the conic

$$5x^2 - 6xy + 5y^2 - 6x - 6y + 5 = 0.$$

Every rational point on this conic will yield a solution of the problem and $a=2b=2c$ is a sample of the only kind of solution which must be discarded as trivial.

The straight line $x=2+lt$, $y=1+mt$, where l and m are rational, will intersect the conic in the point (2, 1) and in another rational point for which the parameter t has the value

$$(8m - 8l)/(5l^2 - 6lm + 5m^2).$$

When we use this value for t, we find the general solution

$$a = 2l^2 - 4lm + 10m^2,$$
$$b = 5l^2 - 14lm + 13m^2,$$
$$c = 5l^2 - 6lm + 5m^2,$$

where l and m are any integers. By inspection, the set (2, 5, 5) is a solution. It is probably the simplest.

Also solved by P. J. Federico, Michael Goldberg, William Hoover, O. J. Ramler, Paul Wernicke, and the Proposer.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Under the auspices of the Edinburgh Mathematical Society, a Mathematical Colloquium will be held in St. Andrews, Scotland, from July 19th to 30th, 1930. The following courses of lectures have been arranged: "Rational curves and surfaces," by H. F. Baker; "Arithmetical properties of curves and surfaces," by H. W. Richmond; "The wave mechanics," by C. G. Darwin; "Elementary mathematics from the higher standpoint," by H. W. Turnbull; "Recent developments in symmetric functions, determinants, and algebraic equations," by A. C. Aitken; "Theory of functions," probably by Professor G. H. Hardy; Informal talks by Professor E. T. Whittaker and others—discussions on the lecture courses.

The fee for the Colloquium (including all the lecture courses) is £1, of which 5s. is payable as the registration fee when application is made. Application should be made to E. T. Copson, Esq., 144 North Street, St. Andrews, as early as possible, and in no case later than June 30th. Early application is particularly advisable in the case of those who propose to stay at the Hall (see below), as this Hostel accommodation is limited.

Members of the Colloquium may stay at the University Hall, which has been reserved entirely for this purpose. The Hostel is divided into three separate wings for ladies, for gentlemen, and for members accompanied by their wives. The cost of board and lodging for the period of the Colloquium (dinner on July 19th to breakfast on July 30th) will be £5 10s. per head; some reduction will be made for shorter periods.

Arrangements will be made for golf, tennis, excursions and other recreations.

Professor Harry W. Tyler, of the Massachusetts Institute of Technology, will retire at the end of the present academic year. Fifty years have lapsed since he entered that institute as a freshman in the course in chemistry at the age of seventeen. After graduation he was an assistant on the staff for two

years, studied a year at Göttingen, took his doctorate in philosophy at Erlangen in 1889, meanwhile having held the grade of instructor on the Institute staff in absentia. In 1899, at the age of twenty-seven, he was made an assistant professor; at twenty-nine, an associate professor; at thirty, a professor: at thirty-eight, head of the department of mathematics; and at forty-seven, Walker Professor of Mathematics. Since its foundation in 1901, he has been active in the work of the College Entrance Examination Board, of which he is now vice-chairman.

Following his retirement from the Institute, Professor Tyler expects to devote his time to furthering the interests of the American Association of University Professors, of which he is general secretary, and as a consultant to the Library of Congress on scientific literature. In so doing he will carry out his wish to follow the line of "maximum usefulness rather than of least resistance," for his retirement is not due to impaired health.

The following courses in mathematics are announced for the summer of 1930:

University of California at Los Angeles, July 1–August 10. By Professor Paul H. Daus: Foundations of arithmetic. By Professor Sophia H. Levy: The teaching of mathematics.

University of Chicago. In addition to the regular courses in differential and integral calculus the following advanced courses are announced for the Summer Quarter, 1930, first term June 16–July 23, second term July 24–August 29. By Professor H. E. Slaught: Differential equations, Theory of definite integrals. By Professor L. E. Dickson: Theory of numbers. By Professor E. P. Lane: Higher plane curves. By Professor E. Bompiani: Analytic projective geometry, Projective differential geometry of hyperspace. By Professor F. D. Murnaghan: Vector analysis, Electrodynamics. By Professor L. M. Graves: Theory of functions of a complex variable, Calculus of variations I. By Professor W. Bartky: Theoretical mechanics II, Modern theories of analytic differential equations II. By Professor Sanger: Solid analytic geometry. By Professor H. S. Everett: Introduction to higher algebra, Algebraic invariants.

Columbia University, July 7–August 15. In addition to courses in trigonometry, solid geometry, analytic geometry, calculus, and methods of teaching secondary mathematics, the following advanced courses are offered: By Professor W. B. Fite: Differential equations. By Professor J. F. Ritt: Functions of a complex variable. By Professor B. O. Koopman: Differential geometry. By Professor P. A. Smith: Fundamental concepts of mathematics. By Dr. M. S. Demos: Introduction to higher algebra.

University of Colorado, first term, June 23–July 26; second term, July 28–August 29. In addition to the usual elementary work in algebra, trigonometry, analytic geometry, and calculus, the following courses will be offered. First term—By Professor Light: Teachers' course in mathematics; History of mathematics; Differential geometry. By Professor Cohen: Differential equations;

Modern algebra; and a graduate course to be arranged in accordance with the desires of the applicants. Second term—By Professor Light: Statistics; Theory of finance; Differential geometry (continued). By Professor Cohen: Theory of equations; Differential equations (continued); Modern algebra (continued).

Cornell University, July 5 to August 15. In addition to the usual elementary work, the following advanced courses will be offered. By Professor J. I. Hutchinson: Higher algebra. By Professor Virgil Snyder: Teachers' course; Projective geometry II. By Professor F. R. Sharpe: Projective geometry I. By Professor W. A. Hurwitz: Theory of functions of a complex variable. By Professor D. C. Gillespie: Advanced calculus. Reading and research work will be directed by Professors J. I. Hutchinson, Virgil Snyder, F. R. Sharpe, W. A. Hurwitz, W. B. Carver, D. C. Gillespie, C. F. Craig, C. F. Roos, and F. R. Bamforth.

University of Illinois, June 16 to August 9. In addition to the usual courses in college algebra, trigonometry, analytic geometry, and calculus, the following advanced courses are offered. By Professor J. B. Shaw: Linear algebra; Modern topics in mathematics. By Professor Arnold Emch: Projective geometry. By Associate Professor A. R. Crathorne: Functions of a complex variable; Statistics. By Assistant Professor H. R. Brahana: Analysis situs. By Dr. L. L. Steimley: Differential equations. By Dr. H. W. Bailey: Fundamental concepts.

University of Iowa, first term, June 9 to July 17. In addition to courses in college algebra, trigonometry, analytic geometry, and calculus, the following subjects are offered. By Miss Ruth Lane: Subject matter and teaching of mathematics. By Professor Weida: Statistics. By Professor Dines: Advanced calculus; Linear inequalities. By Professor Ward: Differential geometry; Differential equations. By Professor Wylie: Celestial mechanics; Descriptive astronomy; Mathematics of finance. By Professor Woods: Modern Geometry; Seminar in geometry. By Professor Reilly: Algebra for high school teachers; Integration and summation; Seminar in interpolation. By the Staff: Reading and research. Second term, July 21 to August 21. By Dr. Conkwright: Differential equations. By Professor Weida: Advanced algebra; Statistics. By Professor Woods: Constructive geometry; Projective geometry. By Professor Chittenden: Advanced calculus; The theory of dimension; Seminar in abstract sets. By the staff: Reading and research.

Johns Hopkins University, June 30 to August 8. By Professor Alonzo Church, of Princeton University: Introduction to Point set theory; College Algebra; Modern geometry (Text, Johnson's *Modern Geometry*).

University of Maine, July 7 to August 15. In addition to the usual elementary work, the following advanced courses are offered. By Associate Professor Bryan: Teachers' course; Higher algebra; Seminar in the teaching of analytic geometry and calculus. By Associate Professor Jordan: Celestial mechanics. By Professor Willard: Advanced calculus; Differential equations; Theory of functions of a complex variable.

Massachusetts Institute of Technology: First period, June 17 to July 29. Courses in calculus and differential equations covering the prescribed work of the first two years. Second period, July 30 to September 10. Courses given in first period repeated. August 11 to September 13. Courses in algebra, solid geometry and trigonometry, in preparation for fall entrance examinations in those subjects. July 7 to August 1. Courses in methods of teaching mathematics in the Junior High School and the Senior High School. June 17 to July 8. Courses in advanced calculus and theoretical aeronautics. July 9 to July 29. Course in theoretical aeronautics continued. July 7 to August 4. Differential equations; intended primarily for army officers.

University of Michigan, June 30 to August 22. In addition to courses in algebra, trigonometry, analytic geometry, elementary calculus, statistics, and finance, the following advanced courses will be offered. By Professor J. W. Bradshaw: Figures of solid geometry; Projective geometry. By Professor P. Field: Vector analysis; Engineering problems. By Professor W. B. Ford: Advanced calculus; Infinite series. By Professor T. H. Hildebrandt: Calculus of variations; Complex Variables. By Professor L. C. Karpinski: Teaching algebra; History of geometry. By Professor T. R. Running: Graphical methods; Empirical formulas. By Professor H. C. Carver: Advanced mathematical theory of statistics. By Professor L. A. Hopkins: Differential equations; Celestial mechanics. By Professor R. L. Wilder: Theory of equations; Foundations of mathematics. By Professor Norman H. Anning: Differential equations. By Professor C. J. Coe: Integral equations. By Professor J. A. Nyswander: Probability; Finite differences.

University of Minnesota, first term, June 17 to July 26. In addition to the usual elementary work, the following courses will be offered. By Professor Dunham Jackson: Advanced algebraic theory. By Professor W. E. Milne, University of Oregon: Differential Equations. By Assistant Professor Gladys Gibbens: Synthetic Metric Geometry. By Professors Jackson, Milne, and Gibbens: Reading in advanced mathematics. Second term, July 28 to August 30. By Professor Raymond W. Brink and Associate Professor Roger A. Johnson: Reading in advanced mathematics.

Northwestern University, June 23 to August 16. In addition to courses in trigonometry, analytic geometry, dfferential and integral calculus, the following advanced courses are offered. By Professor T. F. Holgate: Modern pure geometry. By Professor D. R. Curtiss: Advanced Calculus. By Mr. J. F. Denney: Mathematics of statistics.

Ohio State University, June 17 to August 29. In addition to courses in trigonometry, analytic geometry, calculus, and methods of teaching secondary mathematics the following advanced courses will be offered. By Professor R. E. Langer, of the University of Wisconsin: Partial differential equations; Fourier's Series. By Professor C. C. MacDuffee: Algebraic geometry; linear algebras. By Professor C. T. Bumer: Advanced calculus; vector analysis.

By Professor H. M. Beatty: Advanced Euclidean geometry. Professors Langer and MacDuffee will also direct courses in reading and research.

University of Pennsylvania, July 7 to August 16. In addition to elementary courses, the following advanced courses are offered:— By Professor G. H. Hallett: Theory of finite groups. By Professor F. H. Safford: Partial differential equations. By Professor M. J. Babb; Differential equations. By Professor W. L. G. Williams, of McGill University: Limits and series and theory of numbers.

University of Pittsburgh, June 16 to August 11. In addition to the usual undergraduate courses, the following advanced courses will be offered. By Professor K. D. Swartzel: Functions of a complex variable; Teaching of mathematics. By Professor F. A. Foraker: Modern synthetic geometry; solid analytic geometry. By Associate Professor Taylor: Advanced calculus; functions of a real variable. By Assistant Professor Culver: Differential equations; Theory of equations.

Syracuse University. In addition to the usual courses in mathematics through the calculus the following courses are offered. By Professor F. F. Decker: The teaching of algebra and geometry in secondary schools; Introduction to the theory of invariants or introduction to modern algebra. By Professor A. D. Campbell: Theory of functions of a real variable.

University of Wisconsin, six weeks session, June 30 to August 8. By Professor T. Bennett: Differential equations; College geometry. By Professor H. P. Evans: Vector analysis. By Professor Warren Weaver: Advanced calculus; Theory of probability. Special nine weeks session for graduates, June 30 to August 29. By Professor E. B. Skinner: The Lie theory of differential equations; Theory of finite groups. By Professor H. W. March: Mathematical theory of wave propagation; Partial differential equations of mathematical physics. By Professor L. R. Ingersoll of the Department of Physics: Kinetic theory of gases, with electronic applications; Lectures in selected topics in physical optics; Mathematical theory of heat conduction. (Only one of the two last named courses will be given, the choice depending upon the demand.)

Professor W. O. Beal, of the department of astronomy of the University of Minnesota, died on February 15, 1930, at the age of fifty-six. He was a charter member of the Association.

THE DECEMBER MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION

The twenty-sixth regular meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held at St. John's College, Annapolis, Md., on Saturday, December 7, 1929. Sessions were held in the morning and in the afternoon; Professor W. F. Shenton, Chairman of the Section, presided at both sessions.

Fifty-one persons attended the meeting, including the following thirty-four members of the Association: O. S. Adams, G. F. Alrich, R. N. Ashmun, W. J. Berry, G. A. Bingley, C. C. Bramble, Paul Capron, C. N. Claire, J. L. Clayton, G. R. Clements, Tobias Dantzig, Alexander Dillingham, J. A. Duerksen, J. B. Eppes, P. J. Federico, Michael Goldberg, W. M. Hamilton, F. E. Johnston, L. M. Kells, W. D. Lambert, C. L. Leiper, E. S. Mayer, F. D. Murnaghan, C. H. Rawlins, Jr., H. M. Robert, Jr., R. E. Root, J. B. Scarborough, W. F. Shenton, J. H. Taylor, John Tyler, P. Wernicke, J. Williamson, H. J. Winslow, E. W. Woolard.

During the intermission between the morning and the afternoon sessions, those attending the meeting were entertained at luncheon by the Annapolis members. Following luncheon, there was an interesting and delightful trip through the colonial museum of the Hammand-Harwood house, under the guidance of Professor R. T. H. Halsey of St. Johns College. Preceding the reading of papers at the afternoon session, a vote of thanks was passed in appreciation of the provisions made by St. Johns College for the meeting, and of the exceptionally fine program arranged by the Annapolis members.

The following seven papers were presented:

1. "On the Bertrand paradox," by Professor Tobias Dantzig, University of Maryland.

2. "Solutions of equations by continued fractions," by Professor Paul Capron, U. S. Naval Academy.

3. "Differentiation and substitution," by Professor John Tyler, U. S. Naval Academy.

4. "The rectangular hexagon," by Dr. Paul Wernicke, U. S. Patent Office.

5. "Isometric projection as an aid in solid geometry," by Professor Walter F. Shenton, American University.

6. "Oscillations of a rotating shaft," by Professor R. E. Root, Postgraduate School, U. S. Naval Academy.

7. "Packing of spheres and hyperspheres," by Michael Goldberg, Bureau of Ordnance, Navy Department.

Abstracts of two of these papers follow:

5. The isometric projection of descriptive geometry is the orthogonal projection of a space figure upon a plane with which the three fundamental axes